MA 2161 – Mathematics – II
May / June – 2012 (Regulation 2008)

Part – A

1. Transform the equation $(2x+3)^2\frac{d^2y}{dx^2}-2(2x+3)\frac{dy}{dx}-12y=6x$ into a differential equation with constant coefficients.

Solution:

Substitute $2x+3=e^z$ i.e $2x=e^z-3 \Rightarrow x=\frac{1}{2}(e^z-3)$

$\therefore \log(2x+3)=\log e^z$

$\log(2x+3)=z$

$(2x+3)\frac{dy}{dx}=2\theta y \qquad where\,\theta=\frac{d}{dz}$

$(2x+3)^2\frac{d^2y}{dx^2}=2^2\theta(\theta-1)\,y=(4\theta^2-4\theta)\,y$

Given equation is reduced to

$(4\theta^2-4\theta)\,y-2(2\theta y)-12y=6\left[\frac{1}{2}(e^z-3)\right]$

$(4\theta^2-4\theta-4\theta-12)\,y=3(e^z-3)$

$(4\theta^2-8\theta-12)\,y=3(e^z-3)$

$4(\theta^2-2\theta-3)\,y=3(e^z-3)$

$(\theta^2-2\theta-3)\,y=\frac{3}{4}(e^z-3)$.

2. Find the particular integral of $(D-1)^2\,y=e^x\sin x$.

Solution:

$P.I=\frac{1}{(D-1)^2}e^x\sin x$

Replace D by D+1

$=e^x\frac{1}{(D+1-1)^2}\sin x$

$=e^x\frac{1}{D^2}\sin x$

$=e^x\frac{1}{D}(-\cos x)$

$P.I=-e^x\sin x$

3. Find λ such that $\vec{F}=(3x-2y+z)\vec{i}+(4x+\lambda y-z)\vec{j}+(x-y+2z)\vec{k}$ is solenoidal.

Solution:

Given $\vec{F}=(3x-2y+z)\vec{i}+(4x+\lambda y-z)\vec{j}+(x-y+2z)$ is solenoidal.

i.e $div\,\vec{F}=0$

i.e $\nabla.\vec{F}=0$

$$i.e\ \frac{\partial}{\partial x}(3x-2y+z)+\frac{\partial}{\partial y}(4x+\lambda y-z)+\frac{\partial}{\partial z}(x-y+2z)=0$$

$$3+\lambda+2=0$$

$$5+\lambda=0$$

$$\lambda=-5$$

4. State Gauss Divergence theorem.

Statement:

If S is a closed surface enclosing a region of space with volume V and if $\vec{F}$ is a vector point function with continuous first order partial derivatives in V, then $\iint_S \vec{F}.\vec{ds}=\iiint_V div\vec{F}dv$.

5. State the basic difference between the limit of a function of a real variable and that of a complex variable .

Solution:

In real variable, $x \to x_0$ implies that x approaches x_0 along the x-axis (or) a line parallel to the x-axis.

In complex variables, $z \to z_0$ implies that z approaches z_0 along any path joining the points z and z_0 that lie in the z-plane.

6. Prove that a bilinear transformation has at most two fixed points.

Proof:

The fixed points of a bilinear transformation $z=\frac{az+b}{cz+d}$ is

$$z=\frac{az+b}{cz+d}$$

$$z(cz+d)=az+b$$

$$cz^2+dz=az+b$$

$$cz^2+dz-az-b=0$$

$$cz^2+(d-a)z-b=0$$

This is a quadratic equation in z.

Hence it has atmost two fixed points.

7. Define singular point.

Definition:

A singular point of an analytic function is a point at which the function fails to be analytic.

8. Find the residue of the function $f(z)=\frac{4}{z^3(z-2)}$ at a simple pole.

Solution:

$z=2$ is a simple pole.

$$\left[\operatorname{Re}s\, f(z)\right]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{4}{z^3(z-2)}$$

$$=\frac{4}{2^3}$$

$$\left[\operatorname{Re}s\, f(z)\right]_{z=2} = \frac{1}{2}.$$

9. State the first shifting theorem on Laplace transforms.

Statement:

If $L[f(t)]=F(s)$ then $L[e^{at}f(t)]=F(s-a)$.

10. Verify initial value theorems for $f(t)=1+e^{-t}(\sin t+\cos t)$

Solution:

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s.F(s)$

$$L.H.S = \lim_{t\to 0} f(t)$$

$$= \lim_{t\to 0}\left[1+e^{-t}(\sin t+\cos t)\right]$$

$$=1+1 \qquad \because e^{-0}=e^0=1$$

$$=2 \qquad \sin 0 = 0$$

$$\cos 0 = 1$$

$$R.H.S = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} s\left[\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{s}{s}+\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{1}{s}\right)^2+1}\right]$$

$$1+1 \qquad \because \frac{2}{\infty}=0 \quad \frac{1}{\infty}=0$$

$$R.H.S = 2$$

Part – B

11. a.i) Solve $(D^2+a^2)y=\sec ax$ using the method of variation of parameters.

Solution:

AE is $m^2+a^2=0$

$m^2 = -a^2$

$m = \pm ai$

$m = 0 \pm i\,a$

$$= e^{0x}\left(A\cos ax + B\sin ax\right)$$

$$C.F = A\cos ax + B\sin ax$$

$$A\ f_1\ +\ B\ f_2$$

$$f_1 = \cos ax \qquad f_2 = \sin ax$$

$$f_1^{'} = -a\sin ax \quad f_2^{'} = a\cos ax$$

$$f_1 f_2^{'} - f_1^{'} f_2 = \cos ax\left(a\cos ax\right) - \left(-a\sin ax\right)\sin ax$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a\left(\cos^2 ax + \sin^2 ax\right)$$

$$f_1 f_2^{'} - f_1^{'} f_2 = a \qquad \because \cos^2 ax + \sin^2 ax = 1$$

$$P.I = Pf_1 + Qf_2$$

Where $P = -\int \frac{f_2 X}{f_1 f_2^{'} - f_1^{'} f_2}\,dx$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2}\,dx$$

$$P = -\int \frac{f_2 X}{f_1 f_2^{'} - f_1^{'} f_2}\,dx$$

$$= -\int \frac{\sin ax \cdot \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\int \sin ax \cdot \frac{1}{\cos ax}\,dx \qquad \because \sec ax = \frac{1}{\cos ax}$$

$$= -\frac{1}{a}\int \tan ax\,dx$$

$$= -\frac{1}{a}\ \frac{\log\left(\sec ax\right)}{a}$$

$$P = -\frac{1}{a^2}\log\left(\sec ax\right)$$

$$Pf_1 = -\frac{1}{a^2}\cos ax\ \log\left(\sec ax\right)$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2}\,dx$$

$$= \int \frac{\cos ax \cdot \sec ax}{a}\,dx$$

$$= \frac{1}{a}\int \cos ax \frac{1}{\cos ax}\,dx$$

$$= \frac{1}{a}\int dx$$

$$Q = \frac{x}{a}$$

$$Qf_2 = \frac{x}{a}\sin ax$$

$$\therefore P.I = -\frac{1}{a^2}\cos ax \log(\sec ax) + \frac{x}{a}\sin ax$$

$\therefore$ General solution is

$$y = C.F + P.I$$

$$= A\cos ax + B\sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax \log(\sec ax)$$

11.a.ii) Solve: $(D^2 - 4D + 3)y = e^x \cos 2x$.

Solution:

AE is $m^2 - 4m + 3 = 0$

$$m^2 - m - 3m + 3 = 0$$

$$m(m-1) - 3(m-1) = 0$$

$$(m-3)(m-1) = 0$$

$m - 3 = 0 \qquad m - 1 = 0$

$m = 3 \qquad m = 1$

$m_1 = 3 \qquad m_2 = 1$

$m_1 \neq m_2$

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{3x} + Be^{x}$$

$$P.I = \frac{1}{D^2 - 4D + 3} e^x \cos 2x$$

$$= e^x \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 4D - 4 + 3} \cos 2x$$

$$= e^x \frac{1}{D^2 - 2D} \cos 2x$$

$$= e^x \frac{1}{-4 - 2D} \cos 2x \qquad D^2 = -(2^2) = -4$$

$$= e^x \frac{1}{(-4-2D)} \frac{(-4+2D)}{(-4+2D)} \cos 2x$$

$$= e^x \frac{(-4+2D)}{(-4)^2 - (2D)^2} \cos 2x$$

$$= e^x \frac{(-4\cos 2x + 2D\cos 2x)}{16 - 4D^2}$$

$$= e^{x}\,\frac{\left[-4\cos 2x+2\left(-2\sin 2x\right)\right]}{16-4\left(-4\right)}$$

$$= e^{x}\,\frac{\left(-4\cos 2x-4\sin 2x\right)}{16+16}$$

$$= e^{x}\,\frac{\left(-4\right)\left(\cos 2x+\sin 2x\right)}{32}$$

$$= -\frac{1}{8}e^{x}\left(\cos 2x+\sin 2x\right)$$

$$P.I = \frac{-e^{x}}{8}\left(\sin 2x+\cos 2x\right)$$

$\therefore$ The general solution is

$$y = C.F + P.I$$

$$= A\,e^{3x} + Be^{x} - \frac{e^{x}}{8}\left(\sin 2x+\cos 2x\right)$$

11.b.i) Solve the differential equation $\left(x^2D^2 - xD + 4\right)y = x^2\sin\left(\log x\right)$.

Solution:

Given $\left(x^2D^2 - xD + 4\right)y = x^2\sin\left(\log x\right)$

Put $x = e^{z}$ $\qquad \therefore\ x^2 = \left(e^{z}\right)^2 = e^{2z}$

$\log x = \log e^{z}$

$\log x = z$

$xD = \theta \qquad where\,\theta = \frac{d}{dz}$

$x^2D^2 = \theta\left(\theta-1\right) = \theta^2 - \theta$

$\left(\theta^2 - \theta - \theta + 4\right)y = e^{2z}\sin z$

$\left(\theta^2 - 2\theta + 4\right)y = e^{2z}\sin z$

AE is $\quad m^2 - 2m + 4 = 0$

$$m = \frac{-\left(-2\right)\pm\sqrt{\left(-2\right)^2 - 4\left(1\right)\left(4\right)}}{2\left(1\right)}$$

$$= \frac{2\pm\sqrt{4-16}}{2}$$

$$= \frac{2\pm\sqrt{-12}}{2}$$

$$= \frac{2\pm i\,2\sqrt{3}}{2}$$

$$= \frac{2}{2} \pm i\frac{2\sqrt{3}}{2}$$
$$= 1 \pm i\sqrt{3}$$

$$C.F = e^{z}\left(A\cos\sqrt{3}\,z + B\sin\sqrt{3}z\right)$$

$$P.I = \frac{1}{\theta^2 - 2\theta + 4}e^{2z}\sin z$$

$$= e^{2z}\frac{1}{(\theta+2)^2 - 2(\theta+2) + 4}\sin z \qquad \theta = \theta + 2$$

$$= e^{2z}\frac{1}{\theta^2 + 4\theta + 4 - 2\theta - 4 + 4}\sin z$$

$$= e^{2z}\frac{1}{\theta^2 + 2\theta + 4}\sin z \qquad \theta^2 = -(1)^2 = -1$$

$$= e^{2z}\frac{1}{-1 + 2\theta + 4}\sin z$$

$$= e^{2z}\frac{1}{3 + 2\theta}\sin z$$

$$= e^{2z}\frac{1}{(3+2\theta)} \times \frac{(3-2\theta)}{(3-2\theta)}\sin z$$

$$= e^{2z}\frac{(3-2\theta)}{3^2 - (2\theta)^2}\ \sin z$$

$$= e^{2z}\frac{(3\sin z - 2\theta\sin z)}{9 - 4\theta^2}$$

$$= e^{2z}\frac{(3\sin z - 2\cos z)}{9 - 4(-1)}$$

$$= e^{2z}\frac{(3\sin z - 2\cos z)}{13}$$

` $$P.I = \frac{e^{2z}}{13}(3\sin z - 2\cos z)$$

$\therefore$ The general solution is

$$y = C.F + P.I$$

$$= e^{z}\left(A\cos\sqrt{3}\,z + B\sin\sqrt{3}\,z\right) + \frac{e^{2z}}{13}(3\sin z - 2\cos z)$$

$$= e^{\log x}\left[A\cos\sqrt{3}(\log x) + B\sin\sqrt{3}(\log x)\right] + \frac{e^{2\log x}}{13}\left[3\sin(\log x) - 2\cos(\log x)\right]$$

$$e^{\log x} = x$$
$$e^{2\log x} = e^{\log x^2} = x^2$$

$$= x\left[A\cos\sqrt{3}\left(\log x\right)+B\sin\sqrt{3}\left(\log x\right)\right]+\frac{x^2}{13}\left[3\sin\left(\log x\right)-2\cos\left(\log x\right)\right].$$

11.b.ii) Solve the simultaneous differential equations $\frac{dx}{dt}+2y=\sin 2t$, $\frac{dy}{dt}-2x=\cos 2t$.

Solution:

Given $\frac{dx}{dt}+2y=\sin 2t$ *ie* $Dx+2y=\sin 2t \rightarrow (1)$ *where* $D=\frac{d}{dt}$

$\frac{dy}{dt}-2x=\cos 2t$ *ie* $-2x+Dy=\cos 2t \rightarrow (2)$

$(1)\times 2 \Rightarrow 2Dx+4y=2\sin 2t$

$(2)\times D \Rightarrow \frac{-2Dx+D^2y=D\cos 2t}{4y+D^2y=2\sin 2t+D\cos 2t}$

$D^2y+4y=2\sin 2t-2\sin 2t$

$\left(D^2+4\right)y=0$

$AE\ is\ m^2+4=0$

$m^2=-4$

$m=\sqrt{-4}$

$m=\pm 2i$

$m=0\pm i2$

$=e^{0t}\left(A\cos 2t+B\sin 2t\right)$ $e^{0t}=e^0=1$

$C.F=A\cos 2t+B\sin 2t$

$\therefore y=C.F\ only$ $(\because R.H.S=0)$

$Y=A\cos 2t+B\sin 2t$

Substitute $y=A\cos 2t+B\sin 2t\ in\,(2)$

$-2x+D\left(A\cos 2t+B\sin 2t\right)=\cos 2t$

$-2x+D\left(A\cos 2t\right)+D\left(B\sin 2t\right)=\cos 2t$

$-2x+AD(\cos 2t)+BD\left(\sin 2t\right)=\cos 2t$

$-2x+A\left(-2\sin 2t\right)+B\left(2\cos 2t\right)=\cos 2t$

$-2x-2A\sin 2t+2B\cos 2t=\cos 2t$

` $-2A\sin 2t+2B\cos 2t-\cos 2t=2x$

$\frac{1}{2}\left(-2A\sin 2t+2B\cos 2t-\cos 2t\right)=x$

$\therefore x=-A\sin 2t+B\cos 2t-\frac{1}{2}\cos 2t$

The solution is

$$x=-A\sin 2t+B\cos 2t-\frac{\cos 2t}{2}$$

$$y=A\cos 2t+B\sin 2t$$

12. a.i) Show that $\vec{F}=\left(y^2+2xz^2\right)\vec{i}+(2xy-z)\vec{j}+\left(2x^2z-y+2z\right)\vec{k}$ is irrotational and hence find its scalar potential.

Solution:

Given $\vec{F}=\left(y^2+2xz^2\right)\vec{i}+(2xy-z)\vec{j}+\left(2x^2z-y+2z\right)\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}\left(2x^2z-y+2z\right)-\frac{\partial}{\partial z}(2xy-z)\right]-\vec{j}\left[\frac{\partial}{\partial x}\left(2x^2z-y+2z\right)-\frac{\partial}{\partial z}\left(y^2+2xz^2\right)\right]+\vec{k}\left[\frac{\partial}{\partial x}(2xy-z)-\frac{\partial}{\partial z}\left(y^2+2xz^2\right)\right]$$

$$=\vec{i}(-1+1)-\vec{j}(4xz-4xy)+\vec{k}(2y-2y)$$

$\nabla\times\vec{F}=0 \;\Rightarrow \vec{F}$ is irrotational.

To find scalar potential $\vec{F}=\nabla\phi$

I.e $\left(y^2+2xz^2\right)\vec{i}+(2xy-z)\vec{j}+\left(2x^2z-y+2x\right)\vec{k}=\vec{i}\dfrac{\partial\phi}{\partial x}+\vec{j}\dfrac{\partial\phi}{\partial y}+\vec{k}\dfrac{\partial\phi}{\partial z}$

$\Rightarrow \dfrac{\partial\phi}{\partial x}=y^2+2xz^2$ Integrating w.r.t . x $\phi=xy^2+x^2z^2+f_1(y,z)$

$\dfrac{\partial\phi}{\partial y}=2xy-z$ Integrating w.r.t. y $\phi=xy^2-yz+f_2(x,z)$

$\dfrac{\partial\phi}{\partial z}=2x^2z-y+2z$ Integrating w.r.t. z $\phi=x^2z^2-yz+z^2+f_3(x,y)$

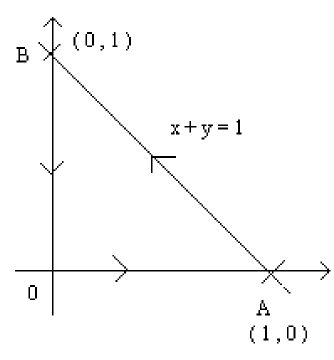
Hence $\phi=xy^2+x^2z^2-yz+z^2+c$, where c is an arbitrary constant.

12.a.ii) Verify Green's theorem in a plane for $\int_C\left[\left(3x^2-8y^2\right)dx+(4y-6xy)dy\right]$ where C is the boundary of the region defined by $x=0, y=0, x+y=1$.

Solution:

Green's theorem is $\int_C(Mdx+Ndy)=\iint_R\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)dxdy$

The given line integral is $\int_C\left[\left(3x^2-8y^2\right)dx+(4y-6xy)dy\right]$



Here $M=3x^2-8y^2$ $\quad N=4y-6xy$

$\therefore\dfrac{\partial M}{\partial y}=-16y$ $\quad \dfrac{\partial N}{\partial x}=-6y$

$\therefore\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}=-6y+16y=10y$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \int_0^1 \int_0^{1-x} 10y\, dydx$$

$$= 10\int_0^1 \left(\frac{y^2}{2} \right)_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2\, dx$$

$$= 5\left[\frac{(1-x)^3}{-3} \right]_0^1$$

$$= -\frac{5}{3}(0-1)$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \quad = \frac{5}{3} \qquad \text{——— (1)}$$

$$\int_C (Mdx + Ndy) = \int_C \left[\left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy \right]$$

$$= \int_{OA} \left[\left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy \right] + \int_{AB} \left[\left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy \right]$$

$$+ \int_{BO} \left[\left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy \right]$$

$$\int_C (Mdx + Ndy) = I_1 + I_2 + I_3 \qquad \text{——— (2)}$$

Along OA

$y = 0 \quad \therefore dy = 0$ and x varies from o to 1.

$$\therefore I_1 = \int_0^1 3x^2 dx$$

$$I_1 = 3\left(\frac{x^3}{3} \right)_0^1$$

$$I_1 = 1 \qquad \text{——— (3)}$$

Along AB

$x + y = 1 \Rightarrow y = 1 - x \quad \therefore dy = -dx$

x varies from 1 to 0

$$\therefore I_2 = \int_1^0 \left\{ \left[3x^2 - 8(1-x)^2 \right] dx + \left[4(1-x) - 6x(1-x) \right](-dx) \right\}$$

$$= \int_1^0 \left[3x^2 - 8(1-x)^2 - 4(1-x) + 6\left(x - x^2\right) \right] dx$$

$$= \left[x^3 - \frac{8(1-x)^3}{-3} - 4\frac{(1-x)^2}{-2} + 6\left(\frac{x^2}{2} - \frac{x^3}{3} \right) \right]_1^0$$

$$= \left(0 + \frac{8}{3} + 2 + 0 \right) - \left[1 + 0 + 0 + 6\left(\frac{1}{2} - \frac{1}{3} \right) \right]$$

$$= \frac{8}{3} + 2 - 1 - 1$$

(4)

$$I_2 = \frac{8}{3}$$ ———— ④

Along BO

$x = 0 \quad \therefore dx = 0$ and y varies from 1 to 0

$$\therefore I_3 = \int_1^0 4y\,dy$$

$$= 4\left(\frac{y^2}{2}\right)_1^0$$

$$I_3 = -2$$ ———— ⑤

Substitute (3),(4) and (5) in (2)

$$\int_C (Mdx + Ndy) = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$ ———— ⑥

From ① and ⑥

$$\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

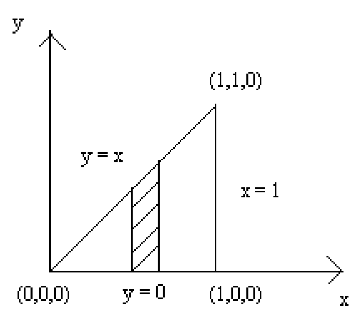
12.b) i) Using Stoke's theorem, evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and 'C' is the boundary of the triangle with vertices at $(0,0,0),(1,0,0),(1,1,0)$.

Solution:

By Stoke's theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\,ds$

The triangle formed by the given points lie on the $xy-plane$. So $\hat{n} = \vec{k}$.

Curl $\vec{F} = \nabla \times \vec{F}$



$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-x-z) - \frac{\partial}{\partial z}(x^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-x-z) - \frac{\partial}{\partial z}(y^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right]$$

$$= \vec{i}(0-0) - \vec{j}(-1-0) + \vec{k}(2x-2y)$$

$$curl\,\vec{F} = \vec{j} + (2x-2y)\vec{k}$$

$$\therefore Curl\,\vec{F} \cdot \hat{n} = \left[\vec{j} + (2x-2y)\vec{k}\right] \cdot \vec{k}$$

$$\therefore Curl\,\vec{F} \cdot \hat{n} = 2x - 2y.$$

$$\therefore \iint_s Curl\,\vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^x 2(x-y)\,dydx$$

$$= 2\int_0^1\int_0^x (x-y)\,dydx$$

$$= 2\int_0^1\left(xy-\frac{y^2}{2}\right)_0^x dydx$$

$$= 2\int_0^1\left(x^2-\frac{x^2}{2}\right)dx$$

$$= \int_0^1 x^2dx$$

$$= \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{1}{3}$$

$$\therefore \int_C \vec{F}\cdot d\hat{r} = \frac{1}{3}.$$

12.b. ii) Find the work done in moving a particle in the force field given by $\vec{F} = 3x^2\vec{i} + (2xz-y)\vec{j} + z\vec{k}$ along the straight line from (0,0,0) to (2,1,3).

Solution:

The equation of a straight line joining (0,0,0) and (2,1,3) is $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t\,(say)$$

$i.e \quad \frac{x}{2} = t \qquad \frac{y}{1} = t \qquad \frac{z}{3} = t$

$\Rightarrow x = 2t \qquad y = t \qquad z = 3t$

$dx = 2dt \qquad dy = dt \qquad dz = 3dt$

When $t=0$ $\quad (2t,\ t,\ 3t)$ becomes (0,0,0,)

When $t=1$ $\quad (2t,\ t,\ 3t)$ becomes (2,1,3)

Work done W = $\int_C \vec{F}\cdot\vec{dr}$

$$\therefore \vec{F}\cdot\vec{dr} = \left[3x^2\vec{i} + (2xz-y)\vec{j} + z\vec{k}\right]\cdot\left(\vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz\right)$$

$$= 3x^2\,dx + (2xz-y)\,dy + zdz$$

$$= 3(2t)^2(2dt) + \left[2(2t)3t - t\right](dt) + (3t)(3dt)$$

$$= 24t^2dt + \left(12t^2 - t\right)dt + 9t\,dt$$

$$= \left(24t^2 + 12t^2 - t + 9t\right)dt$$

$$\vec{F}\cdot\vec{dr} = \left(36t^2 + 8t\right)dt$$

$$\therefore \int_C \overrightarrow{F} \cdot \overrightarrow{dr} = \int_{t=0}^{t=1} \left(36t^2 + 8t\right)$$

$$= \left[36\frac{t^3}{3} + \frac{t^2}{2}\right]_0^1$$

$$= 12 + 4$$

$$= 16\,units$$

13.a. i) Prove that every analytic function $w = u + i\,v$ can be expressed as a function of z alone, not as a function of $\overline{z}$.

Solution:

$$Let\; z = x + i\; y$$

$$\therefore \overline{z} = x - i\; y$$

$$\therefore z + \overline{z} = 2x \qquad\qquad z - \overline{z} = 2i\; y$$

$$\frac{z + \overline{z}}{2} = x \qquad\qquad \frac{z - \overline{z}}{2i} = y$$

$$x = \frac{z + \overline{z}}{2} \qquad\qquad y = \frac{z - \overline{z}}{2i}$$

$$\frac{\partial x}{\partial z} = \frac{1}{2} \qquad \frac{\partial y}{\partial z} = \frac{1}{2i} = -\frac{i}{2} \qquad \because 1 = -i^2$$

$$\frac{\partial x}{\partial \overline{z}} = \frac{1}{2} \qquad \frac{\partial y}{\partial z} = -\frac{1}{2i} = \frac{i}{2}$$

$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\left(u + i\,v\right)$$

$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} + i\frac{\partial v}{\partial \overline{z}} \qquad \text{———} \qquad (1)$$

$$\frac{\partial u}{\partial \overline{z}} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \overline{z}}$$

$$= \frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{i}{2}\right)$$

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) \qquad \text{———} \qquad (2)$$

$$\frac{\partial v}{\partial \overline{z}} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \overline{z}}$$

$$= \frac{\partial v}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial v}{\partial y}\left(\frac{i}{2}\right)$$

$$\frac{\partial v}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right) \qquad \text{———} \qquad (3)$$

Since $f\left(z\right) = u + i\,v$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{By } C - R \text{ equation}$$

$(1) \Rightarrow \dfrac{\partial w}{\partial \bar{z}} = \dfrac{\partial u}{\partial \bar{z}} + i\dfrac{\partial v}{\partial \bar{z}}$

$= \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial u}{\partial y}\right) + i\dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + i\dfrac{\partial v}{\partial y}\right)$ from (2) and (3)

$= \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial x} - \dfrac{\partial v}{\partial y}\right)$

$\dfrac{\partial w}{\partial \bar{z}} = \dfrac{1}{2}\left[\dfrac{\partial u}{\partial x} + i\dfrac{\partial u}{\partial y} - i\dfrac{\partial u}{\partial y} - \dfrac{\partial u}{\partial x}\right]$

$= \dfrac{1}{2}(0)$

$\dfrac{\partial w}{\partial \bar{z}} = 0$

$\Rightarrow$ w does not contain $\bar{z}$

$\Rightarrow$ w can be expressed interms of z alone.

13.a. ii) Find the Bilinear transformation which maps the points $z = 0, 1, \infty$ into $w = i, 1, -i$ respectively.

Solution:

The Bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ into the points $w_1 = i, w_2 = 1, w_3 = -i$ is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$= \frac{(z-z_1)\, z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

i.e

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{(z-0)\left(\frac{1}{\infty}-1\right)}{\left(\frac{z}{\infty}-1\right)(1-0)}$$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{z(-1)}{(-1)(1)} \qquad \because \frac{1}{\infty} = \frac{z}{\infty} = 0$$

$$\frac{w-i}{w+i} = \frac{z(1-i)}{(1+i)}$$

$$= \frac{z(1-i)(1-i)}{(1+i)(1-i)}$$

$$= \frac{z(1-i)^2}{1^2 - i^2}$$

$$\frac{w-i}{w+i} = \frac{z(1-2i+i^2)}{1+1}$$

$$\frac{w-i}{w+i} = \frac{-2iz}{2}$$

$$\frac{w-i}{w+i} = \frac{-iz}{1}$$

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{-iz+1}{-iz-1} \qquad If\ \frac{a}{b} = \frac{c}{d}\ then \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\frac{w-i\ +\ w+i}{w-i\ -\ w-i} = \frac{1-iz}{-(1+iz)}$$

$$\frac{2w}{-2i} = \frac{1-iz}{-(1+iz)}$$

$$w = \frac{i(1-iz)}{(1+iz)}$$

$$w = \frac{z+i}{1+iz}$$

13.b.i) If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$.

Solution:

We know that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$L.H.S = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)|$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left(\log|f(z)|\right)$$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left[\log\left(|f(z)|^2\right)^{1/2}\right]$$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left\{\log\left[f(z)\overline{f(z)}\right]^{1/2}\right\} \qquad \because |f(z)|^2 = f(z)\overline{f(z)}$$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left[\frac{1}{2}\log f(z)\overline{f(z)}\right] \qquad \because \log m^n = n\log m$$

$$= 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\left[\log f(z) + \log\overline{f(z)}\right] \qquad \because \log^{mn} = \log^m + \log^n$$

$$= 2\frac{\partial}{\partial z}\left[0 + \frac{1}{\overline{f(z)}} \cdot \overline{f'(z)}\right]$$
$$= 2\frac{\partial}{\partial z}\left[\overline{\frac{f'(z)}{f(z)}}\right]$$
$$= 2 * 0$$
$$= 0 \qquad = R.H.S$$

13.b.ii) Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscate $r^2 = \cos 2\theta$.

Solution:

Given transformation is $w = \frac{1}{z}$

$$\text{i.e } z = \frac{1}{w}$$
$$\text{i.e } x + iy = \frac{1}{u + iv}$$
$$= \frac{1}{(u+iv)} \times \frac{(u-iv)}{(u-iv)}$$
$$= \frac{u - iv}{u^2 - (iv)^2}$$
$$= \frac{u - iv}{u^2 + v^2}$$
$$x + iy = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$$

$$\Rightarrow \quad x = \frac{u}{u^2 + v^2} \qquad y = \frac{-v}{u^2 + v^2}$$

Equation to the hyperbola is $x^2 - y^2 = 1$

$$\text{i.e. } \left(\frac{u}{u^2 + v^2}\right)^2 - \left(\frac{-v}{u^2 + v^2}\right)^2 = 1$$

$$\frac{u^2}{(u^2 + v^2)^2} - \frac{v^2}{(u^2 + v^2)^2} = 1$$

$$\frac{u^2 - v^2}{(u^2 + v^2)^2} = 1.$$

i.e. $u^2 - v^2 = (u^2 + v^2)^2$ is the image in cartesian co-ordiantes.

Using polar co-ordinates. ie. $u = r\cos\theta$ and $v = r\sin\theta$

$$(r\cos\theta)^2-(r\sin\theta)^2=\left[(r\cos\theta)^2+(r\sin\theta)^2\right]^2$$

$$r^2\cos^2\theta-r^2\sin^2\theta \;=\left(r^2\cos^2\theta+r^2\sin^2\theta\right)^2$$

$$r^2\left(\cos^2\theta-\sin^2\theta\right) \;=\left[r^2\left(\cos^2\theta+\sin^2\theta\right)\right]^2$$

$$r^2\cos 2\theta \;=\left(r^2\right)^2 \quad \because \cos^2\theta-\sin^2\theta=\cos 2\theta$$

$$\cos 2\theta \;=r^2 \qquad\qquad \cos^2\theta+\sin^2\theta=1$$

$\Rightarrow \quad r^2 \quad = \cos 2\theta$ is the lemniscate.

14.a.i) Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ where C is $|z-2|=\frac{1}{2}$ by using Cauchy's integral formula.

Solution:

Given $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ Here a=1,2.

C is $|z-2|=\frac{1}{2}$

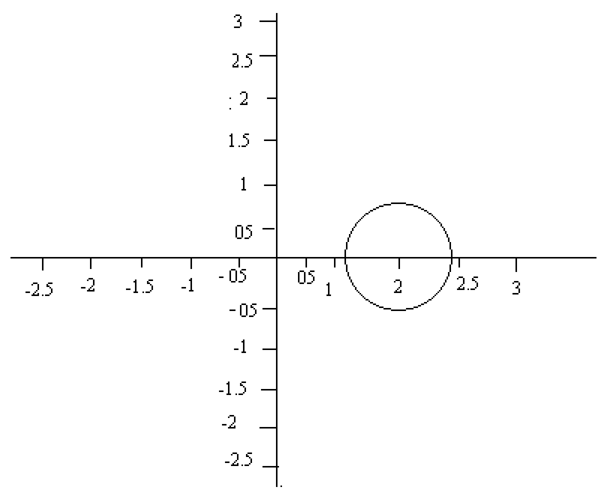
i.e $|x+iy-2|=\frac{1}{2}$

$$\sqrt{(x-2)^2+y^2}=\frac{1}{2}$$

$$(x-2)^2+y^2=\left(\frac{1}{2}\right)^2$$

i.e $(x-2)^2+(y-0)=\left(\frac{1}{2}\right)^2$

Centre $(2,0)$ radius $\frac{1}{2}$



$a=1$ lies outside the circle $|z-2|=\frac{1}{2}$

$a=2$ lies inside the circle $|z-2|=\frac{1}{2}$

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2}=\int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2}dz=2\pi i\; f'(2) \qquad (1)$$

$$f(z)=\frac{z}{z-1}$$

$$f'(z)=\frac{(z-1)1-z(1)}{(z-1)^2}$$

$$=\frac{z-1-z}{(z-1)^2}$$

$$f'(z)\ =\frac{-1}{(z-1)^2}$$

$$f'(2)=-1 \quad \text{------------------} \quad (2)$$

Substitute (2) in (1)

$$\int_C \frac{zdz}{(z-1)(z-2)^2}=2\pi i(-1)$$

$$\int_C \frac{zdz}{(z-1)(z-2)^2}=-2\pi i$$

14.a.ii) Evaluate $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent series valid for the regions $|z|>3$ and $1<|z|<3$

Solution:

Given $f(z)=\frac{1}{(z+1)(z+3)}$

$$\frac{1}{(z+1)(z+3)}=\frac{A}{z+1}+\frac{B}{z+3} \quad \text{------------} \quad (1)$$

$$1 \quad = A(z+3)+B(z+1) \quad \text{--------} \quad (2)$$

Put $z=-1$ in (2)

$1=A(-1+3)+B(-1+1)$

$1=2A+0$

$ie.2A=1$

$A=\frac{1}{2}$

Put $z=-3$ in (2)

$1=A(-3+3)+B(-3+1)$

$1=0-2B$

$ie.-2B=1$

$B=\frac{-1}{2}$

Substitute the values of A and B in (1)

$$\frac{1}{(z+1)(z+3)}=\frac{\frac{1}{2}}{(z+1)}+\frac{\frac{-1}{2}}{(z+3)}$$

i.e. $f(z)=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$

For $|z|>3$

i.e. $3<|z|$

$\frac{3}{|z|}<1$

$$f(z)=\frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$=\frac{1}{2z}\frac{1}{\left(1+\frac{1}{z}\right)}-\frac{1}{2z}\frac{1}{\left(1+\frac{3}{z}\right)}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z)=\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^{2}-\left(\frac{1}{z}\right)^{3}+\cdots\right]-\frac{1}{2z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^{2}-\left(\frac{3}{z}\right)^{3}+\cdots\right]$$

$$=\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{1}{z}\right)^{n}-\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{3}{z}\right)^{n}$$

The above series is valid if $\left|\frac{1}{z}\right|<1$ and $\left|\frac{3}{z}\right|<1$

If $\frac{|1|}{|z|}<1$ and $\frac{|3|}{|z|}<1$

If $\frac{1}{|z|}<1$ and $\frac{3}{|z|}<1$

If $1<|z|$ and $3<|z|$

For $1<|z|<3$

$1<|z|$ $\qquad$ $|z|<3$

$\frac{1}{|z|}<1$ $\qquad$ $\frac{|z|}{3}<1$

$$f(z)=\frac{1}{2}\frac{1}{(z+1)}-\frac{1}{2}\frac{1}{(z+3)}$$

$$=\frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2}\frac{1}{3\left(\frac{z}{3}+1\right)}$$

$$=\frac{1}{2z}\frac{1}{\left(1+\frac{1}{z}\right)}-\frac{1}{6}\frac{1}{\left(1+\frac{z}{3}\right)}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^{2}-\left(\frac{1}{z}\right)^{3}+\cdots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}-\left(\frac{z}{3}\right)^{3}+\cdots\right]$$

$$f(z)=\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n-\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

14.b.i) Evalutate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is the circle $|z-i|=2$ using Cauchy's Residue theorem.

Solution:

Let $f(z)=\frac{z-1}{(z+1)^2(z-2)}$

$z=-1$ is a pole of order 2 ie. double pole

$z=2$ is a pole of order 1. i.e. Simple pole

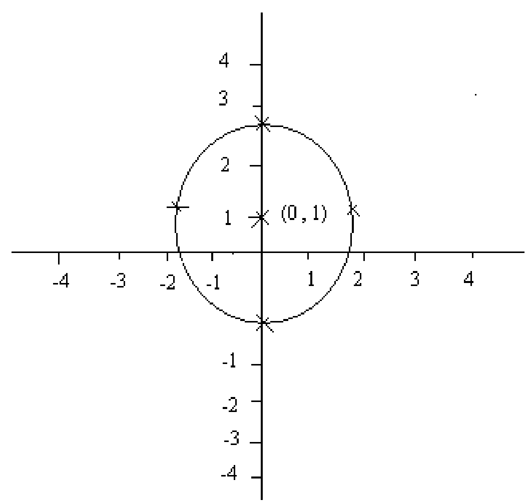
Given
$$|z-i|=2$$
$$|x+iy-i|=2$$
$$|x+i(y-1)|=2$$
$$\sqrt{x^2+(y-1)^2}=2$$
$$x^2+(y-1)^2=2^2$$
$$(x-0)^2+(y-1)^2=2^2$$

Centre (0,1) radius 2

$z=2$ lies outside the circle $|z-i|=2$

$z=-1$ lies inside the circle $|z-i|=2$

$$[\text{Re}sf(z)]_{z=-1}=\lim_{z\to-1}\frac{d}{dz}\left[(z+1)^2f(z)\right]$$
$$=\lim_{z\to-1}\frac{d}{dz}\left[(z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right]$$
$$=\lim_{z\to-1}\frac{d}{dz}\left[\frac{(z-1)}{(z-2)}\right]$$
$$=\lim_{z\to-1}\left[\frac{(z-2).1-(z-1).1}{(z-2)^2}\right]$$
$$=\lim_{z\to-1}\left[\frac{z-2-z+1}{(z-2)^2}\right]$$
$$=\frac{-1}{9}$$



Hence $\sum R=-\frac{1}{9}$

By Cauchy's Residue Theorem $\int_C f(z)\,dz=2\pi i\sum R$.

$$\text{Ie. }\int_C\frac{(z-1)}{(z+1)^2(z-2)}dz=2\pi i\left(\frac{-1}{9}\right)\quad=\frac{-2\pi i}{9}.$$

14.b.ii) Evaluate $\int_0^\infty \frac{\cos mx}{x^2+a^2}dx$ using contour integration.

Solution:

Consider $\int_C \frac{e^{imz}}{z^2+a^2}dz = \int_C f(z)\,dz$, where C is the closed contour consisting of semicircle Γ of radius R. to include all the poles of $f(z)$ and the real axis from $-R$ to R.

$\therefore \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \int_C f(z)\,dz$.

As $R \to \infty \int_\Gamma f(z)\,dz = 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R \qquad\text{———————} \qquad (1)$$

[By Cauchy's Residue theorem $\int_C f(z)\,dz = 2\pi i \sum R$.]

The Poles of $f(z) = \frac{e^{imz}}{z^2+a^2}$ are $z^2+a^2=0 \Rightarrow z = \pm\, ai$

$z = ai$ lies inside the upper half of the semicircle

$$\left[\operatorname{Res} f(z)\right]_{z=ai} = \lim_{z\to ai}(z-ai)f(z)$$

$$\left[\operatorname{Res} f(z)\right]_{z=ai} = \lim_{z\to ai}\left[(z-ai)\frac{e^{imz}}{z^2+a^2}\right]$$

$$= \lim_{z\to ai}\left[(z-ai)\frac{e^{imz}}{z^2-(ia)^2}\right]$$

$$= \lim_{z\to ai}\left[(z-ai)\frac{e^{imz}}{(z+ai)(z-ai)}\right]$$

$$= \frac{e^{im(ai)}}{ai+ai}$$

$$\left[\operatorname{Res} f(z)\right]_{z=ai} = \frac{e^{-ma}}{2ai}$$

$$\therefore \sum R = \frac{e^{-ma}}{2ai}$$

Substitute (2) in (1)

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\frac{e^{-ma}}{2ai}\right)$$

Ie. $\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+a^2}dx = \frac{\pi e^{-ma}}{a}$

$$\int_{-\infty}^{\infty} \frac{(\cos mx + i\sin mx)}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$$

Equating real part on both sides

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$$

Ie. $2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}$ $\qquad \because \frac{\cos mx}{x^2+a^2}$ is an even function.

$$\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{2a}.$$

15.a.i) Apply convolution theorem to evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$$

$$WKT \quad L^{-1}\left[F(s)\cdot G(s)\right] = \int_0^t f(u)\,g(t-u)\,du$$

$F(s) = \frac{s}{s^2+a^2}$ $\qquad G(s) = \frac{1}{s^2+a^2}$

i.e. $L\left[f(t)\right] = \frac{s}{s^2+a^2}$ $\qquad ie \;\; L\left[g(t)\right] = \frac{1}{s^2+a^2}$

$f(t) = L^{-1}\left[\frac{s}{s^2+a^2}\right]$ $\qquad g(t) = L^{-1}\left[\frac{1}{s^2+a^2}\right]$

$f(t) = \cos at$ $\qquad g(t) = \frac{1}{a}\sin at$

$f(u) = \cos au$ $\qquad g(t-u) = \frac{1}{a}\sin a(t-u)$

$$L^{-1}\left[F(s).G(s)\right] = \int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_o^t \cos au\,\frac{1}{a}\sin a(t-u)\,du$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{a}\int_0^t \cos au\,\sin(at-au)\,du\,.$$

$$\cos A\,\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos au\,\sin(at-au) = \frac{1}{2}\left\{\sin\left[au+(at-au)\right] - \sin\left[au-(at-au)\right]\right\}$$

$$= \frac{1}{2}\left[\sin(au+at-au) - \sin(au-at+au)\right]$$

$$= \frac{1}{2}\left[\sin at - \sin(2au-at)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{1}{a}\int_0^t \frac{1}{2}\left[\sin at-\sin\left(2au-at\right)\right]du$$

$$=\frac{1}{2a}\int_0^t\left[\sin at\,du-\sin\left(2au-at\right)du\right]$$

$$=\frac{1}{2a}\left[\int_0^t \sin at\,du-\int_0^t \sin\left(2au-at\right)du\right]$$

$$=\frac{1}{2a}\left[\sin at\int_0^t du-\int_0^t \sin\left(2au-at\right)du\right]$$

$$=\frac{1}{2a}\left\{\sin at\left(u\right)_0^t-\left[\frac{-\cos\left(2au-at\right)}{2a}\right]_0^t\right\}$$

$$=\frac{1}{2a}\left\{t\sin at+\frac{1}{2a}\left[\cos at-\cos\left(-at\right)\right]\right\}$$

$$=\frac{1}{2a}\left[t\sin at+\frac{1}{2a}\left(\cos at-\cos at\right)\right] \qquad \because \cos\left(-\theta\right)=\cos\theta$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t\sin at}{2a}.$$

15.a.ii) Find the Laplace transform of the following triangular wave function given by

$$f\left(t\right)=\begin{cases} t\ , & 0\le t\le\pi \\ 2\pi-t, & \pi\le t\le 2\pi\end{cases} \quad \text{and } f\left(t+2\pi\right)=f\left(t\right).$$

Solution:

The function $f\left(t\right)$ has period 2π $\quad$ *ie.* $T=2\pi$

$$L\left[f\left(t\right)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f\left(t\right)$$

$$=\frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}f\left(t\right)dt$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st} f(t)\,dt+\int_{\pi}^{2\pi} e^{-st} f(t)\,dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st} t\,dt+\int_{\pi}^{2\pi} e^{-st}(2\pi-t)\,dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} t\,e^{-st} dt+\int_{\pi}^{2\pi}(2\pi-t)e^{-st} dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right)-1\left(\frac{e^{-st}}{s^2}\right)\right]_0^{\pi}+\left[(2\pi-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\left(\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}\right)_0^{\pi}+\left[-(2\pi-t)\frac{e^{-st}}{s}+\frac{e^{-st}}{s^2}\right]_{\pi}^{-2\pi}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\left[\left(\frac{-\pi e^{-s\pi}}{s}-\frac{e^{-s\pi}}{s^2}\right)+\frac{1}{s^2}\right]+\left[\left(\frac{e^{-s2\pi}}{s^2}\right)-\left(\frac{-\pi e^{-s\pi}}{s}+\frac{e^{-s\pi}}{s^2}\right)\right]\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left(\frac{-\pi e^{-\pi s}}{s}-\frac{e^{-\pi s}}{s^2}+\frac{1}{s^2}+\frac{e^{-2\pi s}}{s^2}+\frac{\pi e^{-\pi s}}{s}-\frac{e^{-\pi s}}{s^2}\right)$$

$$L[f(t)]=\frac{1}{1-e^{-2\pi s}}\left(\frac{1}{s^2}-\frac{2e^{-\pi s}}{s^2}+\frac{e^{-2\pi s}}{s^2}\right)$$

$$=\frac{1}{1-e^{-2\pi s}}\left(\frac{1-2e^{-\pi s}+e^{-2\pi s}}{s^2}\right)$$

$$=\frac{1}{1^2-\left(e^{-\pi s}\right)^2}\left[\frac{1^2-2(1)e^{-\pi s}+\left(e^{-\pi s}\right)^2}{s^2}\right]$$

$$=\frac{1}{\left(1+e^{-\pi s}\right)\left(1-e^{-\pi s}\right)}\left[\frac{\left(1-e^{-\pi s}\right)^2}{s^2}\right]$$

$$=\frac{1}{s^2}\frac{1-e^{-\pi s}}{1+e^{-\pi s}}$$

$$=\frac{1}{s^2}\tan h\left(\frac{\pi s}{2}\right) \qquad \because \frac{1-e^{-x}}{1+e^{-x}}=\tan\left(\frac{x}{2}\right)$$

15.b.i) Find the Laplace transform of $\frac{e^{at}-e^{-bt}}{t}$

Solution:

Provided $\underset{t\to o}{lt}\frac{f(t)}{t}$ *exists*

WKT $L\left[\frac{f(t)}{t}\right]=\int_s^{\infty} F(s)\,ds$

$$\underset{t \to o}{lt} \frac{l^{at} - e^{-bt}}{t} = 0$$

i.e. $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L\left[f(t)\right] ds$$

$$L\left[\frac{e^{at} - e^{-bt}}{t}\right] = \int_s^\infty L\left(e^{at} - e^{-bt}\right) ds$$

$$= \int_s^\infty \left[L\left(e^{at}\right) - L\left(e^{-bt}\right) ds\right]$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s+b}\right) ds$$

$$L\left[\frac{e^{at} - e^{-bt}}{t}\right] = \left[\log(s-a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{s+b}\right)\right]_s^\infty \qquad \because \log^m - \log^n = \log\left(\frac{m}{n}\right)$$

$$= \left\{\log\left[\frac{s\left(1 - \frac{a}{s}\right)}{s\left(1 + \frac{b}{s}\right)}\right]\right\}_s^\infty$$

$$= \left[\log\left(\frac{1 - \frac{a}{s}}{1 + \frac{b}{s}}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1 - \frac{a}{s}}{1 + \frac{b}{s}}\right)$$

$$= 0 - \log\left(\frac{\frac{s-a}{s}}{\frac{s+b}{s}}\right)$$

$$= -\log\left(\frac{s-a}{s+b}\right)$$

$$= (-1)\log\left(\frac{s-a}{s+b}\right)$$

$$= \log\left(\frac{s-a}{s+b}\right)^{-1}$$

$$L\left[\frac{e^{at}-e^{-bt}}{t}\right]=\log\left(\frac{s+b}{s-a}\right)$$

15.b.ii) Evaluate $\int_0^\infty te^{-2t}\cos t\,dt$ using Laplace transform.

Solution:

$$\int_0^\infty e^{-st}f(t)\,dt=F(s)$$

i.e. $\int_0^\infty e^{-st}f(t)\,dt=L\left[f(t)\right]$

$$\int_0^\infty e^{-2t}\,t\cos t\,dt=\left[L(t\cos t)\right]_{s=2}$$

$$=\left\{-\frac{d}{ds}\left[L(\cos t)\right]\right\}_{s=2}$$

$$=\left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s=2}$$

$$=\left\{-\left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right]\right\}_{s=2}$$

$$=\left[-\frac{(s^2+1-2s^2)}{(s^2+1)^2}\right]_{s=2}$$

$$=\left[\frac{-(-s^2+1)}{(s^2+1)^2}\right]_{s=2}$$

$$=\left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2}$$

$$=\frac{2^2-1}{(2^2+1)^2}$$

$$\int_0^\infty e^{-2t}\,t\cos t\,dt=\frac{3}{25}$$

15.b.ii) Solve the differential equation $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-t}$ with $y(0)=1$ and $y'(0)=0$, using Laplace transform.

Solution:

Given $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-t}$ $\qquad y(0)=1$ and $y'(0)=0$

i.e $y''-3y'+2y=e^{-t}$

Taking the Laplace transform on both sides $L(y'')-3L(y')+2L(y)=L\left(e^{-t}\right)$

$$\left[s^2L(y)-sy(0)-y'(0)\right]-3\left[sL(y)-y(0)\right]+2L(y)=\frac{1}{s+1}$$

$$\left[s^2L(y)-s\cdot 1-0\right]-3\left[sL(y)-1\right]+2L(y)=\frac{1}{s+1}$$

$$s^2L(y)-s-3sL(y)+3+2L(y)=\frac{1}{s+1}$$

$$s^2L(y)-3sL(y)+2L(y)=\frac{1}{s+1}+s-3$$

$$\left(s^2-3s+2\right)L(y)=\frac{1+s^2+s-3s-3}{s+1}$$

$$\left(s^2-3s+2\right)L(y)=\frac{s^2-2s-2}{(s+1)}$$

$$L(y)=\frac{s^2-2s-2}{(s+1)\left(s^2-3s+2\right)}$$

$$L(y)=\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} \longrightarrow \quad (1)$$

$$\frac{s^2-2s-2}{(s-1)(s+1)(s-2)} = \frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s-2} \longrightarrow \quad (2)$$

$$s^2-2s-2 = A(s+1)(s-2)+B(s-1)(s-2)+C(s-1)(s+1) \longrightarrow \quad (3)$$

Put $s=-1$ in (3) $\quad 1=6B \quad \Rightarrow \quad B=\frac{1}{6}$

Put $s=1$ in (3) $\quad -3=-2A \Rightarrow \quad A=\frac{3}{2}$

Put $s=2$ in (3) $\quad -2=3C \quad \Rightarrow \quad C=\frac{-2}{3}$

Substitute the values of A, B and C in (2)

$$\frac{s^2-2s-2}{(s-1)(s+1)(s-2)} = \frac{\frac{3}{2}}{s-1}+\frac{\frac{1}{6}}{s+1}+\frac{\frac{-2}{3}}{s-3}$$

i.e. $$\frac{s^2-2s-s}{(s-1)(s+1)(s-2)} = \frac{3}{2}\frac{1}{s-1}+\frac{1}{6}\frac{1}{s+1}-\frac{2}{3}\frac{1}{s-2} \longrightarrow \quad (4)$$

Substitute (4) in (1)

$$L(y)=\frac{3}{2}\frac{1}{s-1}+\frac{1}{6}\frac{1}{s+1}-\frac{2}{3}\frac{1}{s-2}$$

$$\therefore y=L^{-1}\left[\frac{3}{2}\frac{1}{s-1}+\frac{1}{6}\frac{1}{s+1}-\frac{2}{3}\frac{1}{s-2}\right]$$

$$=\frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right]+\frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right]-\frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$=\frac{3}{2}e^{t}+\frac{1}{6}e^{-t}-\frac{2}{3}e^{2t}$$

MA 2161 – Mathematics – II
Nov / Dec – 2011 (Regulation 2008)

Part – A

1. Transform the equation $x^2 y'' + xy' = x$ into a linear differential equation with constant coefficients.

Solution:

Given $\left(x^2D^2 + xD\right)y = x$ --------------------------------(1)

Take $x = e^z \quad z = \log x$

$xD = D' \quad x^2D^2 = D'(D'-1)$

$(1) \Rightarrow \left(D'(D'-1) + D'\right) y = e^z$

$\Rightarrow \left(D'^2 - D' + D'\right) y = e^z$

$D'^2 \ y = e^z$

2. Find the particular Integral of $\left(D^2 + 4\right) y = \sin 2x$.

Solution:

$$\text{P.I} = \frac{1}{D^2 + 4} \sin 2x$$

$$= \frac{x}{2D} \sin 2x$$

$$= \frac{x}{2} \int \sin 2x \ dx$$

$$= \frac{-x}{2} \left(\frac{\cos 2x}{2}\right)$$

$$\text{PI} = \frac{-x \cos 2x}{4}.$$

3. P.T $\overline{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational .

Solution:

Given $\overline{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

$$\nabla \times \overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\,[0] + \vec{j}\,[0] + \vec{k}\,[0]$$

$\nabla \times \overline{F} = 0.$

Hence $\overrightarrow{F}$ is irrotational.

4. Prove by Green's theorem that the area bounded by a simple closed C curve is $\frac{1}{2}\int_C (x\,dy - y\,dx)$

Solution:

Green's theorem states that $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here $M = -y \quad N = x$

$$\frac{\partial M}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = 1$$

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= 2\iint_R \text{dx dy}$$

$$= 2\left[\text{Area of closed curve}\right]$$

$\therefore$ Area of closed Curve by $C = \frac{1}{2}\int_C x\,dy - y\,dx$.

5. Show that an analytic function with constant imaginary part is constant.

Solution:

Let $f(z) = u + iv$ be analytic.

Then $C - R$ equations are satisfied.

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Given that $v = c$ (constant)

Hence $\frac{\partial v}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$

Thus $\frac{\partial u}{\partial y} = 0 \quad \frac{\partial u}{\partial x} = 0$ [using $C - R$ equations]

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= 0 + i0$

$= 0$

$\Rightarrow f(z)$ is constant.

6. Find the invariant point of the transformation $w = \frac{1+z}{1-z}$.

Solution:

Given $w = \frac{1+z}{1-z}$

Put $w = z$

$z = \frac{1+z}{1-z}$

$\Rightarrow z(1-z) - 1 - z = 0$

$\Rightarrow z^2 + 1 = 0$

$\Rightarrow z = \pm i$

$\therefore$ The invariant points are $i, -i$.

7. Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z| = \frac{1}{2}$

Solution:

$f(z) = \frac{z}{(z-1)(z-2)}$

Here $z = 1$ lies outside C.

$z = 2$ lies outside C.

Hence by Cauchy's integral theorem

$\int_C f(z)\,dz = 0$

$\therefore \int_C \frac{z\,dz}{(z-1)(z-2)} = 0$

8. Calculate the residue of $f(z) = \frac{e^2 z}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^2 z}{(z+1)^2}$

Pole $z + 1 = 0$

$z = -1$

$z = -1$ is a pole of order 2.

$$\left[\text{Res } f(z)\right]_{z=-1} = \lim_{z\to -1} \frac{d}{dz}\left((z+1)^2 \cdot \frac{e^{2z}}{(z+1)^2}\right)$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(e^{2z}\right)$$

$$= \lim_{z\to -1} 2e^{2z}$$

$\text{Res } f(z)\big]_{z=-1} = 2e^{-2}.$

9. Find $L\left(e^{-3t}\sin t\cos t\right)$

Solution:

$L\left(e^{-3t}\sin t\cos t\right) = L(\sin t\cos t)\big]_{s\to s+3}$

$= L\left(\frac{\sin 2t}{2}\right)\Big]_{s\to s+3}$

$$=\frac{1}{2}\left(\frac{2}{s^2+4}\right)_{s\to s+3}$$

$$=\frac{1}{(s+3)^2+4}$$

$$=\frac{1}{s^2+6s+13}.$$

10. Find inverse Laplace transform of $\frac{e^{-as}}{s}$.

Solution:

Wkt $L(u(t-a))=\frac{e^{-as}}{s}$

$$\therefore L^{-1}\left(\frac{e^{-as}}{s}\right)=u(t-a).$$

PART – B

11.a) i) Solve the equation $(D^2-3D+2)y=2\cos(2x+3)+2e^x$.

Solution:

A.E is $m^2-3m+2=0$

$$(m-2)(m-1)=0$$

$$m=2,1$$

i.e., $m_1=2;m_2=1$ [Roots are real & distinct]

$\therefore$ C.F is,

$$y=Ae^{2x}+Be^x$$

$$P.I_1=\frac{1}{D^2-3D+2}\,2\cos(2x+3)$$

$$=\frac{2}{-4-3D+2}\cos(2x+3)$$ Replace D^2 by -4

$$=\frac{2}{-3D-2}\times\frac{-3D+2}{-3D+2}\cos(2x+3)$$

$$=\frac{-6D+4}{(-3D)^2-4}\cos(2x+3)$$

$$=\frac{-6D+4}{9D^2-4}\cos(2x+3)$$

$$=\frac{-6D+4}{-40}\cos(2x+3)$$ [Again D^2 by -4].

$$=\frac{-3D+2}{-20}\cos(2x+3)$$

$$=\frac{-1}{20}\left[-3D\cos(2x+3)+2\cos(2x+3)\right]$$

$$=\frac{-1}{20}\left[6\sin(2x+3)+2\cos(2x+3)\right]$$

$$=\frac{-3}{10}\sin(2x+3)-\frac{1}{10}\cos(2x+3)$$

$$\therefore P.I_2=\frac{2}{D^2-3D+2}e^x$$

$$=\frac{2}{1-3+2}e^x$$ [Replace D by 1] Invalid

$$=\frac{2x}{2D-3}e^x$$

$$=\frac{2x}{2(1)-3}e^x$$ [Again D by 1]

$$P.I._2=-2xe^x$$

$\therefore$ Complete solution is,

$$y=C.F+P.I_1+P.I_2$$

$$=Ae^{2x}+Be^x-\frac{3}{10}\sin(2x+3)-\frac{1}{10}\cos(2x+3)-2xe^x$$

11.a) ii) Apply the method of variation of parameters to solve $(D^2+4)y=\cot 2x$.

Solution:

$$(D^2+4)y=\cot 2x$$

A.E., is, $m^2+4=0$

$$m^2=-4$$

$$m=\pm 2i$$

The roots are imaginary.

$\therefore$ C.F. is

$$y=c_1\cos 2x+c_2\sin 2x$$

(i.e.,) $C.F.=c_1f_1+c_2f_2$

Where $f_1=\cos 2x$; $f_2=\sin 2x$

$$f_1'=-2\sin 2x \; ; \; f_2'=2\cos 2x$$

$$f_1f_2'-f_1'f_2=2\cos^2 2x+2\sin^2 2x=2(1)=2$$

$$P=-\int\frac{f_2X}{f_1f_2'-f_1'f_2}\,dx$$

$$=-\int\frac{\sin 2x.\cot 2x}{2}\,dx$$

$$=-\frac{1}{2}\int\cos 2x\,dx=\frac{-1}{2}\cdot\frac{\sin 2x}{2}$$

$$\therefore \quad P=\frac{-\sin 2x}{4}$$

$$Q=\int \frac{f_1 X}{f_1 f_2^{'}-f_1^{'} f_2}\, dx$$

$$=\frac{1}{2}\int \cos 2x .\cot 2x\, dx$$

$$=\frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\, dx$$

$$=\frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x}\, dx$$

$$=\frac{1}{2}\int \cos ec\, 2x\, dx - \frac{1}{2}\int \sin 2x\, dx$$

$$=\frac{1}{2}\log\left[\cos ec 2x-\cot 2x\right].\frac{1}{2}-\frac{1}{2}.\left(\frac{-\cos 2x}{2}\right)$$

$$=\frac{1}{4}\log\left[\cos ec 2x-\cot 2x\right]+\frac{\cos 2x}{4}$$

$\therefore\ P.I.=Pf_1+Qf_2$

$$=\left(\frac{-\sin 2x}{4}\right)(\cos 2x)+\left(\frac{1}{4}\log\left(\cos ec 2x-\cot 2x\right)+\frac{\cos 2x}{4}\right)(\sin 2x)$$

$$=\frac{1}{4}\log\left(\cos ec 2x-\cot 2x\right).\sin 2x$$

$\therefore C.S.$ is,

$$y=C.F.+P.I.$$

$$y=c_1\cos 2x+c_2\sin 2x+\frac{1}{4}\log\left(\cos ec 2x-\cot 2x\right).\sin 2x.$$

11. b) i) Solve:- $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\left[\log(1+x)\right]$

Solution:

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\left[\log(1+x)\right]$$

Put $1+x=e^z$

$$z=\log(1+x)$$

$$(1+x)^2 D^2=1^2 .D^{'}\left(D^{'}-1\right)$$

$$(1+x)D=1.D^{'}$$

$$\left[D^{'}\left(D^{'}-1\right)+D^{'}+1\right]y=4\cos z$$

$$\left[D^{'2}-D^{'}+D^{'}+1\right]y=4\cos z$$

$$\left[D^{'2}+1\right]y=4\cos z$$

A.E., is, $m^2+1=0$

$$m^2=-1\ ;\ m=\pm i$$

$\therefore$ Roots are imaginary.

$\therefore C.F. = A\cos z + B\sin z$

$$P.I. = \frac{1}{D'^2 + 1} 4\cos z$$
$$= \frac{1}{-1+1} 4\cos z$$
$$= \frac{4z}{2D'} \cos z = 2z\int \cos z\, dz$$
$$= 2z\sin z$$

$\therefore C.S.$ is, $y = C.F. + P.I.$

$y = A\cos z + B\sin z + 2z\sin z$

Where $z = \log(1+x)$.

11. b) ii) Solve $\frac{dx}{dt} - y = t$ & $\frac{dy}{dt} + x = t^2$ given, $x(0) = y(0) = 2$.

Solution:

$Dx - y = t ----------(1)$

$Dy + x = t^2 ----------(2)$

$(1)\times D \Rightarrow D^2x - Dy = 1$

$$\frac{Dy + x = t^2}{Adding\left(D^2+1\right)x = t^2+1}$$

$\left(D^2+1\right)x = t^2+1$

$\therefore C.F.\ is,\quad m^2+1=0\ ;\ m^2=-1\ ;\ m=\pm i$

$C.F. = c_1\cos t + c_2\sin t$

$$\therefore P.I. = \frac{1}{D^2+1} t^2+1$$
$$= \left(1+D^2\right)^{-1}\left(t^2+1\right)$$
$$= \left(1-D^2+D^4-D^6+\cdots\cdots\right)\left(t^2+1\right)$$
$$= 1+t^2-D^2\left(1+t^2\right)+0$$
$$= 1+t^2-2 = t^2-1$$

$\therefore x = c_1\cos t + c_2\sin t + t^2 - 1$

$$\frac{dx}{dt} = -c_1\sin t + c_2\cos t + 2t$$

(1) $\Rightarrow -c_1\sin t + c_2\cos t + 2t - y = t$

$y = -c_1\sin t + c_2\cos t + t$

$\therefore x(t) = -c_1\cos t + c_2\sin t + t^2 - 1$

$x(0) = c_1 - 1 = 2$

$\therefore c_1 = 3$

$\therefore y(t) = -c_1\sin t + c_2\cos t + t$

$y(0) = c_2 = 2$

$$\therefore c_2 = 2 .$$

Solution is,

$$x = 3\cos t + 2\sin t + t^2 - 1$$

$$y = -3\sin t + 2\cos t + t .$$

12.a) Evaluate $\int_C \left(x^2 + xy\right)dx + \left(x^2 + y^2\right)dy$ where C is the square bounded by the lines $x = 0, x = 1, y = 0 \ and \ y = 1$.

<u>Solution:</u>

By Green's theorem,

$$\int_C \left(Mdx + Ndy\right) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

For RHS.

Here $M = x^2 + xy$ $\qquad N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$$

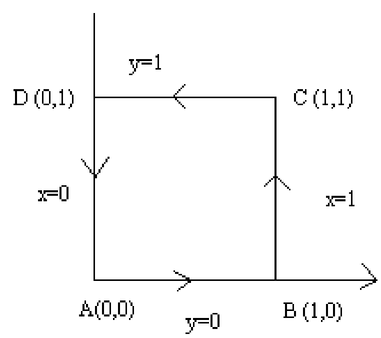
Given $x = 0 \ to \ 1$ $\qquad y = 0 \ to \ 1$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1\int_0^1 x dx\,dy$$

$$= \int_0^1 \left.\frac{x^2}{2}\right)_0^1 dy = \frac{1}{2}\int_0^1 dy$$

$$= \left.\frac{y}{2}\right]_0^1 = \frac{1}{2}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{1}{2} \text{-------------} (1)$$



<u>LHS</u>

Consider

$$\int_C M\,dx + N\,dy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{OA}$$

Along AB, $y = 0, \ dy = 0 \quad x = 0 \ to \ 1$

$$\int_{AB} Mdx + Ndy = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Along BC, $x = 1, \ dx = 0, \quad y = 0 \ to \ 1$

$$\int_{BC} Mdx + Ndy = \int_0^1 \left(1 + y^2\right)dy = \left[y + \frac{y^3}{3}\right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}$$

Along CD, $y=1,\ dy=0,\quad x=1\ to\ 0$

$$\int_{CD} Mdx+Ndy=\int_1^0 \left(x^2+x\right)dx=\left.\frac{x^3}{3}+\frac{x^2}{2}\right]_1^0=-\left[\frac{1}{3}+\frac{1}{2}\right]=\frac{-5}{6}$$

Along OA, $x=0,\ dx=0,\quad y=1\ to\ 0$

$$\int_{OA} Mdx+Ndy=\int_1^0 y^2dy=\left.\frac{y^3}{3}\right]_1^0=-\frac{1}{3}$$

$$\int_C Mdx+Ndy=\frac{1}{3}+\frac{4}{3}-\frac{5}{6}-\frac{1}{3}=\frac{8-5}{6}=\frac{3}{6}=\frac{1}{2}-------(2)$$

From (1) & (2),

$$LHS=RHS=\frac{1}{2}$$

$$\int_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

Hence Green's theorem is verified.

12. b) Verify Gauss Divergence theorem for the vector function $f=\left(x^3-yz\right)\vec{i}-2x^2y\,\vec{j}+2\vec{k}$ over the cube bounded by $x=a,\ y=a,\ z=a.$ and coordinate planes.

Solution:-

Given $\overline{F}=\left(x^3-yz\right)\bar{i}-2x^2y\,\bar{j}+2\bar{k}$

By divergence theorem we have $\iiint_V div\,\overline{F}\,dv=\iint_S \overline{F}.\bar{n}\,ds$

Now $div\,\overline{F}=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\left(\left(x^3-yz\right)\bar{i}-2x^2y\,\bar{j}+2\bar{k}\right)$

$$=\frac{\partial}{\partial x}\left(x^3-yz\right)+\frac{\partial}{\partial y}\left(-2x^2y\right)+\frac{\partial}{\partial z}(2)$$

$$=3x^2-2x^2=x^2$$

$$\iiint_V div\,\overline{F}.dv=\int_0^a\int_0^a\int_0^a x^2dz\,dy\,dx$$

$$=\int_0^a\int_0^a x^2\left[z\right]_0^a dy\,dx$$

$$=\int_0^a\int_0^a x^2\,a\,dy\,dx=a\int_0^a \left.x^2y\right]_0^a dx$$

$$=a\int_0^a a\,x^2dx=a^2\left[\frac{x^3}{3}\right]_0^a=\frac{a^5}{3}.$$

$$LHS=\frac{a^5}{3}-----------(1)$$

Consider,

$$\iint_S \overline{F}\cdot\overline{n}\,ds = \iint_{S_1} \overline{F}\cdot\overline{n}\,ds + \iint_{S_2} \overline{F}\cdot\overline{n}\,ds + \iint_{S_3} \overline{F}\cdot\overline{n}\,ds + \iint_{S_4} \overline{F}\cdot\overline{n}\,ds + \iint_{S_5} \overline{F}\cdot\overline{n}\,ds + \iint_{S_6} \overline{F}\cdot\overline{n}\,ds$$

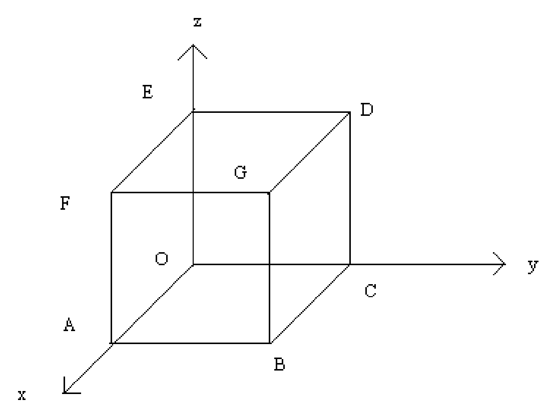
The surface S consists of six faces, the unit normal and the surface element on these faces are tabulated below in order to evaluate the surface integral.

Surface	Equation	$\overline{n}$	ds
ABGF (s_1)	$x=a$	$\overline{i}$	dydz
OCDE (s_2)	$x=0$	$-\overline{i}$	dydz
BCDG (s_3)	$y=a$	$\overline{j}$	dxdz
OAFE (s_4)	$y=0$	$-\overline{j}$	dxdz
DEFG (s_5)	$z=a$	$\overline{k}$	dxdy
OABC (s_6)	$z=0$	$-\overline{k}$	dxdy

<u>*On* S_1,</u>

$$\overline{n}=\overline{i},\ x=a,\ ds=dydz$$

$$\overline{F}.\hat{n}=x^3-yz=a^3-yz$$



$$\iint_{S_1}\overline{F}.\overline{n}\,ds=\int_0^a\int_0^a\left(x^3-yz\right)dy\,dz$$

$$=\int_0^a\int_0^a\left(a^3-yz\right)dy\,dz$$

$$=\int_0^a\left(a^3y-\frac{y^2}{2}z\right)_0^a dz$$

$$=\int_0^a\left(a^4-\frac{a^2}{2}z\right)dz$$

$$=a^4z-\frac{a^2}{2}\frac{z^2}{2}\Big)_0^a$$

$$\iint_{S_1}\overline{F}.\overline{n}\,ds=a^5-\frac{a^4}{4}$$

<u>*On* S_2,</u>

$$\bar{n} = -\bar{i}, \ x = 0, \ ds = dydz$$
$$\bar{F}.\bar{n} = -\left(x^3 - yz\right) = yz - x^3 = yz$$

$$\iint_{S_2} \bar{F}.\bar{n}\, ds = \int_0^a \int_0^a (yz)\, dy\, dz$$

$$= \int_0^a \frac{y^2}{2} z \Big]_0^a dz \quad = \frac{a^2}{2}\frac{z^2}{2}\Big]_0^a$$

$$\iint_{S_2} \bar{F}.\bar{n}\, ds = \frac{a^4}{4}.$$

$On \quad S_3,$

$$\hat{n} = \bar{j}, \ y = a, \ ds = dxdz$$
$$\bar{F}.\bar{n} = -2x^2 y = -2x^2 a$$

$$\iint_{S_3} \bar{F}.\bar{n}\, ds = \int_0^a \int_0^a -2x^2 a \ dxdz$$

$$= \int_0^a -2a\frac{x^3}{3}\Big]_0^a dz$$

$$= \int_0^a \left(\frac{-2}{3} a . a^3\right) dz$$

$$= \frac{-2}{3} a^4 \ z\Big]_0^a = \frac{-2a^5}{3}.$$

$On \quad S_4,$

$$\bar{n} = -\bar{j}, \ y = 0, \ dy = dxdz$$
$$\iint_{S_4} \bar{F}.\bar{n}\, ds = 0$$

$On \quad S_5,$

$$\bar{n} = \bar{k}, \ z = a, \ ds = dxdy$$
$$\iint_{S_5} \bar{F}.\bar{n}\, ds = \int_0^a \int_0^a 2\, dx\, dy = 2a^2$$

$On \quad S_6,$

$$\bar{n} = -\bar{k}, \ z = 0, \ ds = dxdy$$
$$\iint_{S6} \bar{F}.\bar{n}\, ds = \int_0^a \int_0^a -2\, dx\, dy = -2a^2$$

$$\iint_{S_6} \bar{F}.\bar{n}\, ds = a^5 - \frac{a^4}{4} + \frac{a^4}{4} - \frac{2a^5}{3} + 2a^2 - 2a^2$$

$$= a^5 - \frac{2a^5}{3} = \frac{a^5}{3} ----------(2)$$

From (1) and (2) $\iiint_V div\,\overline{F}.dv = \iint_S \left(\overline{F}.\overline{n}\right) ds$

13.a) i) Find the analytic function $w = u + iv$ when $v = e^{-2y}\left(y\cos 2x + x\sin 2x\right)$ and find u

Solution:

Given that $V = e^{-2y}\left(y\cos 2x + x\sin 2x\right)$

$$V_x = e^{-2y}\left[-2y\sin 2x + 2x\cos 2x + \sin 2x\right]$$

$$V_y = -2e^{-2y}\left[y\cos 2x + x\sin 2x\right] + e^{-2y}\left[\cos 2x\right]$$

$$V_x(z,0) = e^0\left[-2(0) + 2z\cos 2z + \sin 2z\right]$$
$$= 2z\cos 2z + \sin 2z$$

$$V_y(z,0) = -2e^0\left[0 + z\sin 2z\right] + \cos 2z$$
$$= z\sin 2z + \cos 2z$$

$$f'(z) = V_y(z,0) + iV_x(z,0)$$
$$= \left(z\sin 2z + \cos 2z\right) + i\left(2z\cos 2z + \sin 2z\right)$$

$$f(z) = \int \left(z\sin 2z + \cos 2z\right) + i\left(2z\cos 2z + \sin 2z\right) dz$$
$$= \int \left[\left(z\sin 2z + i2z\cos 2z\right) + \left(\cos 2z + i\sin 2z\right)\right] dz$$
$$= \int z\left(\sin 2z + i\cos 2z\right) dz + \int \left(\cos 2z + i\sin 2z\right) dz$$

$$\int uv\,dv = uv_1 - u'v_2 + u''v_3 - - - - -$$

$u = z$; $v = \sin 2z + i2\cos 2z$

$u' = 1$; $v_1 = \dfrac{-\cos 2z}{2} + \dfrac{i2\sin 2z}{2}$

$u'' = 0$; $v_2 = \dfrac{-\sin 2z}{4} - \dfrac{i2\cos 2z}{4}$

$$\int z\left(\sin 2z + i2\cos 2z\right) dz = z\left[\frac{-\cos 2z + 2i\sin 2z}{2}\right] + \frac{\sin 2z + i2\cos 2z}{4}$$

So $f(z) = z\left[\dfrac{-\cos 2z + 2i\sin 2z}{2}\right] + \dfrac{\sin 2z + 2i\cos 2z}{4} + \dfrac{\sin 2z - i\cos 2z}{2}$

$$= z\left[\frac{-\cos 2z + 2i\sin 2z}{2}\right] + \frac{\sin 2z + 2i\cos 2z + 2\sin 2z - 2i\cos 2z}{2}$$
$$= z\left[\frac{-\cos 2z + 2i\sin 2z}{2}\right] + \frac{3\sin 2z}{4}$$
$$= \frac{-z\cos 2z}{2} + iz\sin 2z + \frac{3}{4}\sin 2z$$
$$= \frac{3\sin 2z - 2z\cos 2z}{4} + iz\sin 2z$$

13. a) ii) Show that the map $w=\frac{1}{z}$ maps the totality of circles and straight lines as circles or straight lines

Solution:

Given $w=\frac{1}{z}$.

(i.e) $z=\frac{1}{w}$

Now $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{1}{u+iv}\frac{u-iv}{u-iv}=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

$$x=\frac{u}{u^2+v^2}---(1)\ ,\ y=\frac{-v}{u^2+v^2}---(2)$$

The general equation is ,

$$a\left(x^2+y^2\right)+2gx+2fy+c=0-------(3)$$

$$a\left[\frac{u^2}{\left(u^2+v^2\right)^2}+\frac{v^2}{\left(u^2+v^2\right)^2}\right]+2g\left[\frac{u}{u^2+v^2}\right]+2f\left[\frac{-v}{u^2+v^2}\right]+c=0$$

$$a\left[\frac{u^2+v^2}{\left(u^2+v^2\right)^2}\right]+2g\frac{u}{u^2+v^2}-2f\frac{v}{u^2+v^2}+c=0.$$

The transformed equation is,

$$c\left(u^2+v^2\right)+2gu-2fv+a=0$$

i. $a\neq 0, c\neq 0 \Rightarrow$ Circles not passing through the origin in z -plane map into circles not passing through the origin.

ii. $a\neq 0, c=0 \Rightarrow$ Circles through the origin in z-plane map onto straight lines not through the origin.

iii. $a=0, c\neq 0 \Rightarrow$ the straight lines not through the origin in z-plane map onto circles through the origin in w-plane.

iv. $a=0, c=0 \Rightarrow$ straight lines through the origin of z-plane onto straight lines through the origin in w-plane.

13. b) i) If $u(x,y)$ and $v(x,y)$ are harmonic functions in region R prove that the function $\left[\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right]+i\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]$ is an analytic function of $z=x+iy$.

Solution:

As u and v are harmonic, the following are true in R.

i. $U_{xx}+U_{yy}=0$

ii. $V_{xx} + V_{yy} = 0$

iii. Second order partial derivatives of u and v are continuous

Let $U = u_y - v_x$ *and* $V = u_x + v_y$

Then $U_x = u_{xy} - v_{xx}, V_x = u_{xx} + v_{xy}$

$U_y = u_{yy} - v_{yx}$, $V_y = u_{yx} + v_{yy}$

$u_x = v_y$ if $v_{xx} + v_{yy} = 0$ which is true by (i)

$u_y = -v_x$ if $u_{xx} + u_{yy} = 0$ which is true by (ii)

Further u_x, u_y, v_x, v_y are continuous in R by (iii). Hence by sufficiency conditions of analyticity $U + iV$ is an analytic function of z.

13.b) ii) Prove that the transformation $w = \frac{1}{z}$ maps the family of circles and straight lines into the family of circles or straight lines.

Solution:

Refer. 13. a) ii).

14. a) i) Using Cauchy's Integral formula, Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$ where 'C' is the circle $|z+1+i| = 2$

Solution:

$|z+1+i| = 2$ is the circle with centre at $z = -1-i$ & Radius '2'.

$\therefore \frac{z+4}{z^2+2z+5} = \frac{z+4}{(z+1+2i)(z+1-2i)}$

i.e., $z = -1-2i$ lies inside the circle.

$$\int_C \frac{\frac{z+4}{z+1-2i}}{z+1+2} dz = \int_C \frac{f(z)}{z-(-1-2i)} dz, where\ f(z) = \frac{z+4}{z+1-2i}$$

By Cauchy's Integral formula,

$$\int_C \frac{z+4}{z^2+2z+5} dz = 2\pi i\, f(-1-2i) = \frac{\pi}{2}(3-2i)$$

14. a) ii) Find the Laurent's series of $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ valid in $2 < |z| < 3$

Solution:

$f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$\therefore f(z) = A + \frac{B}{z+2} + \frac{C}{z+3}$

$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2)$

Put $z = -2$

$3 = B \Rightarrow B = 3$

Put $z = -3$

$8 = -C \Rightarrow C = -8$

Equating the co-efficient of z^2,

$1 = A \Rightarrow A = 1$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

When $2 < |z| < 3$,

$$2 < |Z| \Rightarrow \left|\frac{2}{Z}\right| < 1$$

$$|Z| < 3 \Rightarrow \left|\frac{Z}{3}\right| < 1$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right]$$

14. b) i) Evaluate using Cauchy's Residue theorem, $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where $C: |z| = 3$

Solution:

Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The poles are $z = 1$ & $z = 2$ which are simple poles.

Both the poles lies inside $C: |z| = 3$

$$\therefore \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i \left[\text{Re}\,sidue\,1 + \text{Re}\,sidue\,2\right]$$

Residue 1 $= \text{Re}\,s\left[f(z)\right]_{z \to +1} = \underset{z\to 1}{lt} (z-1)\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

$$= \frac{-1}{1-2} = 1 \left[\because \sin n\pi = 0\right]$$

Residue 2 $= \text{Re}\,s\left[f(z)\right]_{z \to +2} = \underset{z\to 2}{lt} (z-2)\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

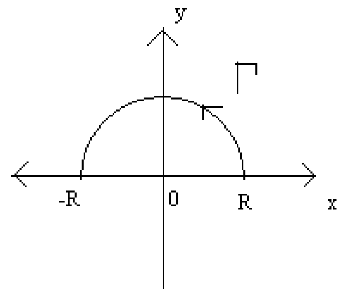
$$= \frac{1}{1} = 1 \left[\because \cos n\pi = (-1)^n\right]$$

$$\therefore \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz = 2\pi i \times [+1+1]$$

$$= 4\pi i.$$

14. b) ii) Evaluate using contour integration $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$.

Solution:



Let $f(z) = \frac{x^2}{(x^2+1)^2} dx$

Where C consists of the semi circle $F : |z| = R$

and the bounding diameter $[-R, R]$

Now $\int_C f(z) dz = \int_{-R}^{R} f(x) dx + \int_\Gamma f(z) dz$

The poles of $f(z)$ are

$z^2 + 1 = 0 \qquad (Twice)$

$z^2 = -1$

$z = \pm i$

$z = i$ is double pole lies inside Γ

$z = -i$ is double pole lies outside Γ

$$\operatorname{Res}[f(z), i] = \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 . f(z)\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 . \frac{z^2}{(z+i)^2 (z-i)^2}\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right]$$

$$= \lim_{z \to i} \frac{(z+i)^2 (2z) - z^2 . 2(z+i)}{(z+i)^4}$$

$$= \frac{(2i)^2 (2(i)) - 2(i)^2 (2i)}{(2i)^4}$$

$$= \frac{8i^3 - 4i^3}{16 i^4} \qquad (i^3 = -i)$$

$$= \frac{4i^3}{16 i^4}$$

$$= \frac{-4i}{16}$$

$$= \frac{-i}{4} \qquad or \quad \frac{1}{4i}$$

Hence by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{2}$$

by Contour integration,

$$\int_{-R}^{R} \frac{x^2}{\left(x^2+1\right)^2}dx + \int_{\Gamma} \frac{z^2}{\left(z^2+1\right)^2}\, dz = \frac{\pi}{2}$$

Applying $\underset{R\to\infty}{Lt}$ on both sides

$$\int_{-\infty}^{\infty} \frac{x^2}{\left(x^2+1\right)}\, dx + 0 = \frac{\pi}{2}\left[\because \int \frac{x^2}{\left(x^2+1\right)}\, dz \to 0 \text{ as } R\to\infty\right]$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{\left(x^2+1\right)^2}\, dx = \frac{\pi}{2}.$$

15. a) i) Find $L^{-1}\left\{\frac{1}{s\left(s^2+4\right)}\right\}$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right] = L^{-1}\left[\frac{1}{s}\right] * \frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= 1 * \frac{1}{2}\sin 2t$$

$$= \frac{1}{2}\int_0^t \sin 2(t-u)\,du$$

$$= \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t$$

$$= \frac{1}{4}\left[\cos 2(t-t) - \cos 2(t-0)\right]$$

$$= \frac{1}{4}\left[1-\cos 2t\right]$$

ii) Find the Laplace transform of a square wave function given by

$$f(t) = \begin{cases} E & for \quad 0 \le t \le \frac{a}{2} \\ -E & for \quad \frac{a}{2} \le t \le a \end{cases} \text{ and } f(t+a) = f(t)$$

Solution:

Since this is a periodic function with period a, $L\left[f(t)\right]=\frac{1}{1-e^{-sa}}\int_0^a e^{-st}f(t)\,dt$

$$\int_0^a e^{-st}f(t)\,dt=\int_0^{\frac{a}{2}} e^{-st}1\,dt+\int_{\frac{a}{2}}^{a} e^{-st}(-1)\,dt$$

$$=1\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{s}}-1\left[\frac{e^{-st}}{-s}\right]_{\frac{a}{s}}^{a}$$

$$=1\left[\frac{1-e^{\frac{-sa}{2}}}{s}\right]-1\left[\frac{e^{\frac{-sa}{2}}-e^{-sa}}{s}\right]$$

$$=\frac{1}{s}\left[1-2e^{\frac{-sa}{2}}+e^{-sa}\right]$$

$$=\frac{1}{s}\left(1-e^{\frac{-sa}{2}}\right)^2$$

$$\therefore L\left[f(t)\right]=\frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{1-e^{-sa}}=\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s\left(1+e^{\frac{-sa}{2}}\right)\left(1-e^{\frac{-sa}{2}}\right)}$$

$$=\frac{1}{s}\frac{1-e^{\frac{-sa}{2}}}{1+e^{\frac{-sa}{2}}}$$

$$=\frac{1}{s}\frac{e^{\frac{sa}{4}}-e^{\frac{-sa}{4}}}{e^{\frac{sa}{4}}+e^{\frac{-sa}{4}}}=\frac{1}{s}\tanh\left(\frac{sa}{4}\right)$$

15. b) i) Evaluate $\int_0^\infty te^{-2t}\cos t\,dt$ using Laplace transforms.

Solution:

$$L[\cos t]=\frac{s}{s^2+1}$$

$$L[t\cos t]=-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$\int_0^\infty te^{-2t}\cos t\,dt=\left[L(t\cos t)\right]_{s=2}$$

$$=\left[-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]\right]$$

$$=\left\{-\left[\frac{(s^2+1)(1)-2s^2}{(s^2+1)^2}\right]\right\}_{s=2}$$

$$=\left\{-\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]\right\}_{s=2} \quad =\left\{-\left[\frac{-s^2+1}{(s^2+1)^2}\right]\right\}_{s=2}$$

$$=\left\{\frac{s^2-1}{(s^2+1)^2}\right\}_{s=2} \quad =\frac{4-1}{(4+1)^2}=\frac{3}{25}$$

15. b)ii) $\frac{d^{2y}}{dt^2}+4\frac{dy}{dt}+4y=\sin t$ if $\frac{dy}{dt}=0$ & $y=2$ when $t=0$ using L.T

Solution:

Given:- $y''+4y'+4y=\sin t$

Taking L.T on both sides, $L(y''(t))+4L(y'(t))=L(\sin t)$

$$s^2\bar{y}-sy(0)-y'(0)-4(s\bar{y}-sy(0))+4\bar{y}=\frac{1}{s^2+1}$$

$$\bar{y}(s^2+4s+4)-2s-8=\frac{1}{s^2+1}$$

$$\bar{y}(s^2+4s+4)=\frac{1}{s^2+1}+2s+8$$

$$=\frac{1+2s^3+2s+8s^2+8}{s^2+1}$$

$$=\frac{2s^3+8s^2+2s+9}{s^2+1}$$

$$\bar{y}=\frac{2s^3+8s^2+2s+9}{(s^2+1)(s^2+4s+4)}$$

$$L(y(t))=\frac{2s^3+8s^2+2s+9}{(s^2+1)(s+2)^2}$$

$$\frac{2s^3+8s^2+2s+9}{(s^2+1)(s+2)^2}=\frac{As+B}{s^2+1}+\frac{C}{s+2}+\frac{D}{(s+2)^2}$$

Put $s=-2$

$$-16+32-4+9=5D$$

$$21=5D$$

$$D=\frac{21}{5}$$

Co-efficient of s^3 is,

$$2 = A + C -------(1)$$

Co-efficient of s^2

$$8 = 4A + B + 2C + D --------(2)$$

Co-efficient of s,

$$4A + 4B + C = 2 ----------(3)$$

Comparing constant terms,

$$9 = 4B + 2C + D ----------(4)$$

Substitute $D = \frac{21}{5}$ in (2)

$$4A + 4B + 2C = 8 - \frac{21}{5}$$

$$4A + B + 2C = \frac{19}{5} ----------(5)$$

(2) $\Rightarrow$ $4A + 4B + C = 2$

$$4A + B + 2C = \frac{19}{5}$$

$$3B - C = \frac{-19}{5} + 2$$

$$3B - C = \frac{-9}{5} --------------(6)$$

Put $D = \frac{21}{5}$ in (4)

$$4B + 2C = 9 - \frac{21}{5}$$

$$4B + 2C = \frac{24}{5} --------------(7)$$

$$(6) \times 2 \Rightarrow 6B - 2C = \frac{-18}{5}$$

$$4B + 2C = \frac{24}{5}$$

$$10B = \frac{6}{5} \quad B = \frac{6}{10 \times 5}$$

$$B = \frac{3}{25}$$

Substitute $B = \frac{3}{25}$ & $D = \frac{21}{5}$ in -----------(4)

$$4B + 2C + D = 9$$

$$4\left(\frac{3}{25}\right) + 2C + \frac{21}{5} = 9$$

$$\frac{12}{25} + 2C + \frac{21}{5} = 9$$

$$2C = 9 - \frac{12}{25} - \frac{21}{5}$$

$$= \frac{225 - 12 - 105}{25} = \frac{108}{25}$$

$$C = \frac{54}{25}$$

Substitute C in ----- (1)

$$A + C = 2$$

$$A + \frac{54}{25} = 2 \qquad \Rightarrow A = 2 - \frac{54}{25} = \frac{-4}{25}$$

$$A = \frac{-4}{25}$$

$$\therefore L[y(t)] = \frac{\frac{-4}{2s}s + \frac{3}{25}}{s^2+1} + \frac{54}{25(s+2)} + \frac{21}{5(s+2)^2}$$

$$L[y(t)] = \frac{-4s+3}{25(s^2+1)} + \frac{54}{25(s+2)} + \frac{21}{5(s+2)^2}$$

$$y(t) = L^{-1}\left[\frac{-4s+3}{25(s^2+1)}\right] + \frac{54}{25}L^{-1}\left[\frac{1}{s+2}\right] + \frac{21}{5}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{25}L^{-1}\left[\frac{-4s+3}{s^2+1}\right] + \frac{54}{25}e^{-2t} + \frac{21}{5}e^{-2t}\,.\,t$$

$$= \frac{1}{25}\left[-4L^{-1}\left(\frac{s}{s^2+1}\right) + 3L^{-1}\left(\frac{1}{s^2+1}\right)\right] + \frac{54}{25}e^{-2t} + \frac{21}{5}te^{-2t}$$

$$= \frac{1}{25}[-4\cos t + 3\sin t] + \frac{54}{25}e^{-2t} + \frac{21}{5}te^{-2t}$$

$$= \frac{-4}{25}\cos t + \frac{3}{25}\sin t + \frac{54}{25}e^{-2t} + \frac{21}{5}te^{-2t}.$$

MA 2161 – Mathematics – II
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Question Paper No. 11484

Part – A

1. Find the Wronskian of y_1, y_2 of $y'' - 2y' + y = e^x \log x$.

Solution :

$$\left(D^2 - 2D + 1\right) y = e^x \log x$$

To find C.F, $m^2 - 2m + 1 - 0$

$$(m-1)^2 = 0 \Rightarrow m = 1,1$$

$\therefore C.F \quad y = \left(C_1 x + C_2\right)e^x = C_1 x e^x + C_2 e^x$

$\Rightarrow y_1 = xe^x,\ y_2 = e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_2^1 & y_2^1 \end{vmatrix} = \begin{vmatrix} xe^x & e^x \\ xe^x + e^x.1 & e^x \end{vmatrix} = xe^{2x} - xe^{2x} - e^{2x}$$

$= \left[-e^{2x}\right] = -e^{2x}$

$\therefore W = -e^{2x}$.

2. Find the particular integral of $\left(D^2 - 4D + 4\right) y = 2^x$

Solution :

$$P.I. = \frac{2x}{D^2 - 4D + 4} = \frac{e^{\log 2x}}{D^2 - 4D + 4} = \frac{e^{x\log 2}}{D^2 - 4D + 4} = \frac{e^{x\log 2}}{(D-2)^2}$$

$(D \rightarrow \log 2)$

$$\therefore P.I = \frac{e^{x\log 2}}{(\log 2 - 2)^2}$$

$$\therefore P.I = \frac{2^x}{(\log 2 - 2)^2}$$

3. Prove that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

Solution :

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ yz & zx & xy \end{vmatrix} = \vec{i}\left[x - x\right] - \vec{j}\left[y - y\right] + \vec{k}\left[z - z\right]$$

$curl\,\vec{F} = \vec{0}$

$\therefore \vec{F}$ is irrotational.

4. State Gauss divergence theorem.

Solution :

If S is a closed surface enclosing a region of space with volume V and if $\vec{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint_S \vec{F}.\vec{ds} = \iiint_v \left(\nabla.\vec{F}\right) dv .$$

5. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Solution :

$$f(z) = \bar{z} = x - iy$$

$u = x$	$v = -y$	$\Rightarrow u_x \neq v_y$
$u_x = 1$	$v_x = 0$	$u_y = -v_x$
$u_y = 0$	$v_y = -1$	

C – R equations are not satisfied at any point or anywhere. $\therefore$ the given function is nowhere differentiable.

6. Find the map of the circle $|z| = 3$ under the transformation $w = 2z$.

Solution :

$$w = 2z$$
$$i.e. w = 2(x + iy)$$
$$\Rightarrow u = 2x,\ v = 2y$$

$|z| = 3$ maps into $u^2 + v^2 = 36$ $\qquad$ $i.e, |w| = 6$

7. Evaluate $\int \frac{zdz}{(z-1)(z-2)}$, where C is the circle $|z| = \frac{1}{2}$.

Solution :

Given $C : |z| = \frac{1}{2}$, by Cauchy's Integral formula, $\int_c f(z)dz = 0$ $[\because z = 1, 2$ points lies outside]

$$\therefore \int_C \frac{zdz}{(z-1)(z-2)} = 0$$

8. If $f(z) = \frac{-1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + ...\right]$, find the residue of $f(z)$ at $z = 1$.

Solution :

Residue of $f(z)$ at $z = 1$

= the coefficient of $\frac{1}{z-1}$ in the Laurent's series of $f(z)$ about $(z = 1) = -1$.

9. Is the linearity property applicable to $L\left[\frac{1-\cos t}{t}\right]$? Reason out.

Solution :

Linearity property is not applicable to $L\left[\frac{1-\cos t}{t}\right]$ since $L\left[\frac{\cos t}{t}\right]$ does not exist.

10. Find the inverse Laplace Transform of $\frac{1}{(s+1)(s+2)}$.

Solution :

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\Rightarrow 1 = A(s+2) + B(s+1)$$

When $s=-2,\ B=-1$

When $s=-1,\ A=1$

$$\therefore \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$$

Part – B

11. a) i) Solve $(D^2+5D+4)y = e^{-x}\sin 2x$.

Solution :

The Auxilliary equation is $m^2+5m+4=0$

$(m+4)(m+1)=0 \Rightarrow m=-1,-4$

$\therefore C.F.$ is $Ae^{-4x}+Be^{-x}$

$$P.I. = \frac{1}{D^2+5D+4}e^{-x}\sin 2x \quad = e^{-x}\frac{1}{(D-1)^2+5(D-1)+4}\sin 2x$$

$$= e^{-x}\frac{1}{D^2-2D+1+5D-5+4}\sin 2x \quad = e^{-x}\frac{1}{D^2+3D}\sin 2x \quad = e^{-x}\frac{1}{3D-4}\sin 2x$$

$$= e^{-x}\frac{3D+4}{(3D+4)(3D-4)}\sin 2x \quad = e^{-x}\frac{3D(\sin 2x)+4\sin 2x}{(3D)^2-4^2}$$

$$= e^{-x}\frac{6\cos 2x+4\sin 2x}{9D^2-16} \quad = e^{-x}\left(\frac{6\cos 2x+4\sin 2x}{-52}\right)$$

$$= \frac{-e^{-x}}{26}\left[3\cos 2x+2\sin 2x\right]$$

$$\therefore y = C.F.+P.I = Ae^{-4x}+Be^{-x}-\frac{e^{-x}}{26}(3\cos 2x+2\sin 2x)$$

11. a) ii) Solve $\frac{d^2y}{dx^2}+y=\cos ecx$ by the method of variation of parameters.

Solution :

The Auxilliary equation is $m^2+1=0$

$m^2=-1;\ m=\pm i$

$\therefore C.F.$ is $A\cos x+B\sin x$

$\therefore y_1=\cos x \qquad\qquad y_2=\sin x$

$y_1=-\sin x \qquad\qquad y_2=\cos x$

$y_1y_2^1-y_2y_1^1=\cos x(\cos x)-\sin x(-\sin x)=\cos^2 x+\sin^2 x=1$

The Particular integral is given by,

$$u=-\int\frac{y_2x}{y_1y_2^1-y_2y_1^1}dx \quad =-\int\frac{\sin x\cos ecx}{1}dx=-\int dx=-x$$

$$v=\int\frac{y_1x}{y_1y_2^1-y_2y_1^1}dx \quad =\int\frac{\cos x-\cos ecx}{1}dx \quad =\int\cot x dx=\log(\sin x)$$

$\therefore P.I.=uy_1+vy_2=-x\cos x+\log(\sin x)\sin x$

$\therefore$ Solution is $y=C.F.+P.I.$

$y=A\cos x+B\sin x-x\cos x+\sin x\log(\sin x)$

11. b) i) Solve $\frac{dx}{dt}+y=e^t, x-\frac{dy}{dt}=t$.

Solution :

$Dx+y=e^t \quad\rightarrow(1)$

$x-Dy=t \quad\rightarrow(2)$

Differentiating (1), we get

$D^2x+Dy=e^t \quad\rightarrow(3)$

Adding (2) and (3), we get

$D^2x+Dy=e^t$

$x-Dy=t$

$(D^2+1)x=e^t+t$

A.E. is $m^2+1=0;\ m=\pm i$

C.F. $x=A\cos t+B\sin t$

$P.I_1=\frac{e^t}{1+D^2}$

$=\frac{e^t}{2}$

$P.I_2=\frac{1}{1+D^2}(t)$

$=(1+D^2)^{-1}(t)$

$=(1-D^2)t=t$

$\therefore \; x = A\cos t + B\sin t + \frac{e^t}{2} + t \quad \rightarrow (4)$

From (1) $y = e^t - Dx \; = e^t - \left[A(-\sin t) + B\cos t + \frac{e^t}{2} + 1 \right]$

$y = A\sin t - B\cos t + \frac{e^t}{2} - 1 \quad \rightarrow (5)$

Hence (4) and (5) gives the solution

11. b) ii) Solve the equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Solution :

$\frac{x^2 d^2 y}{dx^2} + x\frac{dy}{dx} = 12\log x \quad \rightarrow (1)$

Put $x = e^z; \;\; z = \log x$

$x\frac{dy}{dx} = \theta y; \;\; x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y$

$\therefore$ Equation (1) becomes,

$\left[\theta^2 - \theta + \theta\right] y = 12z = \left(\theta^2 - \theta + \theta\right) y = 12z \;\; = \theta^2 y = 12z$

A.E. is, $m^2 = 0, \; m = 0$ (twice)

C.F. is $(Az + B)e^{oz} = (Az + B) = A(\log x) + B$

$P.I. = \frac{12z}{\theta^2} \;\; = 12\frac{1}{\theta}\left[\int zdz\right] \;\; = 12\frac{1}{\theta}\left[\frac{z^2}{2}\right]$

$= 12\int \frac{z^2}{2}dz \;\; = 6\left(\frac{z^3}{3}\right) = 2z^3 = 2(\log x)^3$

$\therefore y = C.F. + P.I$

$y = A(\log x) + B + 2(\log x)^3$

12. a) i) Show that $\vec{F} = (2xy - z^2)\vec{L} + (x^2 + 2yz)\vec{J} + (y^2 - 2zx)\vec{k}$ is irrotational and find its scalar potential.

Solution :

$$\nabla x F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \partial xy - z^2 & x^2 + 2yz & y^2 - 2xz \end{vmatrix}$$

$= \vec{i}[2y - 2y] - \vec{j}[-2z + 2z] + \vec{k}[2x - 2x]$

$\nabla x\vec{F} = 0$

$\Rightarrow \vec{F}$ is irrotational

$\Rightarrow \vec{F} = \nabla\phi$

$\nabla\phi = \left(2xy - z^2\right)\vec{i} + \left(x^2 + 2yz\right)\vec{j} + \left(y^2 - 2xz\right)\vec{k}$

$\sum \vec{i}\dfrac{\partial\phi}{\partial x} = \left(2xy - z^2\right)\vec{i} + \left(x^2 + 2yz\right)\vec{j} + \left(y^2 - 2xz\right)\vec{k}$

$\dfrac{\partial\phi}{\partial x} = 2xy - z^2 \qquad \Rightarrow \phi = x^2y - xz^2 + f_1(y,z)$

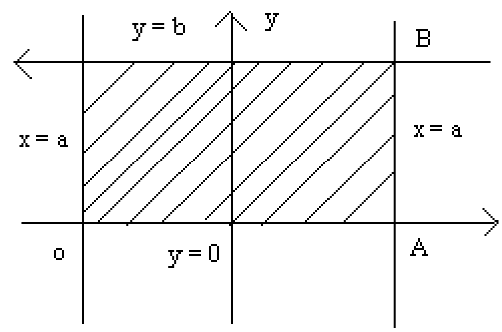
$\dfrac{\partial\phi}{\partial y} = x^2 + 2yz \qquad \Rightarrow \phi = x^2y + y^2z + f_2(x,z)$

$\dfrac{\partial\phi}{\partial z} = y^2 - 2xz \qquad \Rightarrow \phi = y^2z - xz^2 + f_3(x,y)$

$\therefore \phi = x^2y + y^2z - xz^2 + C$.

12. a) ii) Verify Green's theorem for $\vec{v} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0$ and $y = b$.

Solution : Greens Theorem



$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}\right)dxdy$$

$M = x^2 + y^2; \qquad N = -2xy$

$\dfrac{\partial M}{\partial y} = 2y; \quad \dfrac{\partial N}{\partial x} = -2y$

$\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = -2y - 2y = -4y$

R.H.S. $= \iint_R (-4y)\,dxdy = -4\iint ydxdy = -4\int_0^b\int_{-a}^a ydxdy$

$= -4\int_0^b y\left[x\right]_{-a}^a dy = -4\int_0^b 2aydy = -8a\left[\dfrac{y^2}{2}\right]_0^b$

$= -4ab^2 \qquad \to (1)$

L.H.S. $= \int Mdx + Ndy$

$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} Mdx + Ndy$

OA	AB	BC	CO
y=0	x=a	y=b	x=−a
dy=0	dx=0	dy=0	dx=0

$$= \int_{-a}^{a} x^2dx + \int_0^b -2aydy + \int_a^{-a}\left(x^2 + b^2\right)dx + \int_b^0 2aydy$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} - 2a\left(\frac{y^2}{2}\right)_0^b + \left(\frac{x^3}{3}\right)_a^{-a} + b^2(x)_a^{-a} + 2ay\left(\frac{y^2}{2}\right)_b^0$$

$$= \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$= -4ab^2 \qquad \rightarrow (2)$$

From (1) & (2) we get

L.H.S. = R.H.S.

Hence Greens theorem is verified.

12) b) Verify Gauss divergence theorem for $\overrightarrow{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded $x=0, x=1, y=0, y=1, z=0$ and $z=1$.

Solution : Gauss divergence theorem

$$\int\int_S \overrightarrow{F}\cdot\hat{n}ds = \iiint_V \nabla\cdot\overrightarrow{F}dv$$

$$\overrightarrow{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\overrightarrow{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= 4z - 2y + y = 4z - y$$

$$\int\int_S \overrightarrow{F}\cdot\hat{n}ds = \left(\int\int_{S_1} + \int\int_{S_2} + \int\int_{S_3} + \int\int_{S_4} + \int\int_{S_5} + \int\int_{S_6}\right)\overrightarrow{F}\cdot\hat{n}ds$$

$$\int\int_{s_1} \overrightarrow{F}\cdot\hat{n}ds = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}\\ ds=dydz}} 4xzdydz = 0$$

$$\int\int_{s_2} \overrightarrow{F}\cdot\hat{n}ds = \iint_{\substack{x=1\\ \hat{n}=\vec{i}\\ ds=dydz}} 4xzdydz = 0 = \int_0^1\int_0^1 4zdydz = 4\int_0^1\left(\frac{z^2}{2}\right)_0^1 dy = 2\int_0^1 dy = 2\left(y\right)_0^1 = 2$$

$$\int\int_{s_3} \overrightarrow{F}\cdot\hat{n}ds = \iint_{\substack{y=0\\ \hat{n}=-\vec{j}\\ ds=dxdz}} y^2dxdz = 0$$

$$\int\int_{s_4} \overrightarrow{F}\cdot\hat{n}ds = -\iint_{\substack{y=1\\ \hat{n}=-\vec{j}\\ ds=dxdz}} y^2dxdz = -\int_0^1\int_0^1 dxdz = -\int_0^1 (z)_0^1 dx = -\int_0^1 dx = -(x)_0^1 = -1$$

$$\int\int_{s_5} \overrightarrow{F}\cdot\hat{n}ds = -\iint_{\substack{z=0\\ \hat{n}=-\vec{k}\\ ds=dxdy}} yzdxdy = 0$$

$$\int\int_{s_6} \overrightarrow{F}\cdot\hat{n}ds = \iint_{\substack{z=1\\ \hat{n}=\vec{k}\\ ds=dxdy}} yzdydz = \int_0^1\int_0^1 ydydz = \int_0^1\left(\frac{y}{2}\right)_0^1 dz = \frac{1}{2}\int_0^1 dz = \frac{1}{2}(z)_0^1 = \frac{1}{2}$$

$\therefore$ L.H.S. = $\iint_s \vec{F}\cdot\hat{n}ds = 0+2+0-1+\frac{1}{2} = \frac{3}{2} \quad \to (1)$

R.H. S. = $\iiint_v \nabla.\vec{F}dV$

$$= \int_0^1\int_0^1\int_0^1 (4z-y)\,dzdxdy = \int_0^1\int_0^1 \left(2z^2-yz\right)_0^1 dydx = \int_0^1\int_0^1 (2-y)\,dydx$$

$$= \int_0^1\left(2y-\frac{y^2}{2}\right)_0^1 dx = \int_0^1\left(2-\frac{1}{2}\right)dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}[x]_0^1$$

$$= \frac{3}{2} \quad \to (2)$$

From (1) and (2) L.H.S. = R.H.S

Gauss divergence theorem is verified.

13. a) i) Find the bilinear transformation that maps the points $z=\infty, i, 0$ onto $w = 0, i, \infty$

Solution : Given

$z_1 = \infty;\ z_2 = i,\ z_3 = 0$

$w_1 = 0;\ w_2 = i,\ w_3 = \infty$

The bilinear transformation is given by $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_1)}{(z-z_3)(z_2-z_1)}$

$\because z_1 = \alpha,\ w_3 = \infty$

$\therefore \frac{w-0}{i-0} = \frac{i-0}{z-0}$

$\frac{w}{i} = \frac{i}{z}$

$w = -\frac{1}{z}$ which is the bilinear transformation.

a) ii) Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

Solution : $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2x}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2\left(1-\cos^2 2z\right)}{(1-\cos 2z)^2} = \frac{2(1-\cos 2z)(\cos 2z - 1 - \cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = \frac{-1}{\frac{1-\cos 2z}{2}} = \frac{-1}{\sin^2 z} = -\cos ec^2 z$$

$$\phi_2(x,y)=\frac{\partial u}{\partial y}=\frac{(\cosh 2y-\cos 2x)(0)-\sin 2x(2\sinh 2y)}{(\cosh 2y-\cos 2x)^2}$$

$$=\frac{-2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}$$

$$\phi_2(z,0)=0$$

By Milne's Thomson method

$$f(z)=\int\phi_1(z,0)dz-i\int\phi_2(z,0)dz$$

$$=-\int\cos ec^2 z dz-0=\cot z+c$$

$$f(z)=\cot z+c$$

13. b) i) Find the image of the hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$.

Solution : Given $w=\frac{1}{z}$

$$z=\frac{1}{w}$$

$$x+iy=\frac{1}{\mathrm{Re}^{i\phi}}=\frac{1}{R}e^{-i\phi}=\frac{1}{R}[\cos\phi-i\sin\phi]$$

$$x=\frac{1}{R}\cos\phi \qquad y=\frac{1}{R}\sin\phi$$

The image of the hyperbola $x^2-y^2=1$ is $\frac{1}{R^2}\cos^2\phi-\frac{1}{R^2}\sin^2\phi=1$

$$\Rightarrow\frac{1}{R^2}\left[\cos^2\phi-\sin^2\phi\right]=1 \qquad \Rightarrow\frac{1}{R^2}[\cos 2\phi=1] \quad \Rightarrow R^2=\cos 2\phi$$

b) ii) Prove that the transformation $w=\frac{z}{1-z}$ maps the upper half of z-plane into the upper half of w-plane. What is the image of the circle $|z|=1$ under this transformation.

Solution : $w=\frac{z}{1-z} \Rightarrow w(1-z)=z$

$$\Rightarrow z=\frac{w}{1-w}$$

$$x+iy=\frac{u+iv}{1+u+iv}=\frac{u+iv}{1+u+iv}\,\frac{1+u-iv}{1+u-iv}=\frac{\left[u(1+u)+v^2\right]+i\left[v(1+u)-uv\right]}{(1+u)^2+v^2}$$

Separating the imaginary part

$$y=\frac{v}{(1+u)^2+v^2} \qquad \text{when } y>0, v>0$$

i.e., upper half of z-plane maps as upper half of w-plane.

Next, $|z|=1\Rightarrow\left|\frac{w}{1-w}\right|=1\Rightarrow|w|=|1-w|$

$$u^2+v^2=(1+u)^2+v^2$$
$$u^2+v^2=1+2u+u^2+v^2$$
$$\Rightarrow 1+2u=0$$
$\Rightarrow u=-\frac{1}{2}$. Thus $|z|=1$ maps as a line.

14. a) i) Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$, where C is the circle $|z+1+i|=2$ using Cauchy's integral formula.

Solution : Singular points of *f(z)* are

$$z^2+2z+5=0$$
$$z=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm 2i$$

$z=-1+2i$ and $z=-1-2i$

$$\int_C \frac{z+4}{z^2+2z+5}dz=\int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}dz$$

Given, circle is $|z+1+i|=2$

When $z=-1+2i,\ \ |z+1+i|=|-1+2i+1+i|=|3i|=\sqrt{9}=3>2$.

$\therefore$ The point $z=-1+2i$ lies outside C.

When $z=-1-2i,\ \ |z+1+i|=|-1-2i+1+i|=|-i|=1<2$

$\therefore$ The point $z=-1-2i$ lies within C.

$$\therefore \int_C \frac{z+4}{z2+2z+5}dz=\int_C \frac{\frac{z+4}{z-(-1+2i)}}{z-(-1-2i)}dz$$

$$=2\pi i\ f(-1-2i) \text{ where } f(z)=\frac{z+4}{z-(-1+2i)}$$

$$=2\pi i\left(\frac{3-2i}{-4i}\right) \quad \text{since } f(-1-2i)=\frac{-1-2i+4}{-1-2i+1-2i}=\frac{3-2i}{-4i}$$

$$=\frac{\pi}{2}(2i-3).$$

14. a) ii)Find the residues of $f(z)=\frac{z^2}{(z-1)^2(z+2)^2}$ at its isolated singularities using Laurent's series expansions. Also state the valid region.

Solution : $f(z)=\frac{z^2}{(z-1)^2(z+2)^2}$

$$=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z+2}+\frac{D}{(z+2)^2}$$

$$z^2=A(z-1)(z+2)^2+B(z+2)^2+C(z-1)^2(z+2)+D(z-1)^2$$

Put $z=1, \quad B=\frac{1}{9}$

Put $z=-2, \quad D=\frac{4}{9}$

Put $z=0 \qquad 0=-4A+4B+2C+D$

$$0=-4A+\frac{4}{9}+2C+\frac{4}{9}$$

$$0=-4A+\frac{8}{9}+2C$$

$$-4A+2C=-\frac{8}{9} \quad \to (1)$$

Put $z=-1 \qquad 1=-2A+B+4C+4D$

$$1=-2A+\frac{1}{9}+4C+\frac{16}{9}$$

$$-2A+4C=-\frac{8}{9} \quad \to (2)$$

$(1)-(2)\times 2 \Rightarrow -6C=\frac{8}{9} \quad C=-\frac{4}{27}$

$(1) \Rightarrow -4A+2\left(-\frac{4}{27}\right)=-\frac{8}{9}$

$$-4A-\frac{8}{27}=-\frac{8}{9}$$

$$4A=\frac{16}{27} \Rightarrow A=\frac{4}{27}$$

Laurent's series expansion

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n+\sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$\operatorname{Res}\{f(z)\}_{z=4}$=Coefficient b_1 in Laurent's series.

$$f(z)=\frac{4}{27}\frac{1}{z-1}+\frac{1}{9}\frac{1}{(z-1)^2}-\frac{4}{27}\frac{1}{z+2}+\frac{4}{9}\frac{1}{(z+2)^2}$$

At $z=1, \quad b_1=\frac{4}{27}$

At $z=-2, \quad b_1=-\frac{4}{27}$

$\therefore$ Residues are $\frac{4}{27}$ at $z=1$, $|z-1|<3$ and $-\frac{4}{27}$ at $z=-2$, $|z+2|<3$

Region $0<|z-1|<3$ and $0<|z+2|<3$

(OR)

14. b) Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta,\ a > b > 0$

Solution : $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2(a+b\cos\theta)} d\theta$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2a+2b\cos\theta} d\theta$$

$$= \text{R.P. of} \int_0^{2\pi} \frac{1-e^{2i\theta}}{2a+2b\cos\theta} d\theta \qquad \left[\because e^{2i\theta} = \cos 2\theta + i\sin\theta\right]$$

Put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz} \qquad \therefore \cos\theta = \frac{z^2+1}{2z}$

$$\therefore I = R.P.of \int_C \frac{1-z^2}{2a+2b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

where C is the unit circle $|z| = 1$

$$= \text{R.P. of} \int_C \frac{1-z^2}{i\left(bz^2+2az+b\right)} dz$$

$$= \text{R.P. of } \frac{1}{i} \int_C \frac{1-z^2}{bz^2+2az+b} dz$$

$$= \text{R.P. of } \frac{1}{i} \int_C f(z) dz \text{ where } f(z) = \frac{1-z^2}{bz^2+2az+b} \qquad \rightarrow (1)$$

To find residues :

Poles are $z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$

Let $\alpha = \frac{-a+\sqrt{a^2+b^2}}{b},\ \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

α & β are simple poles.

Given $a > b$

$\therefore |\alpha| < 1$ & $|\beta| > 1$

$\therefore \alpha$ lies inside C & β lies outside C.

$$\therefore \operatorname{Res}\{f(z)\}_{z=\alpha} = \lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z\to\alpha} (z-\alpha) \frac{1-z^2}{b(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1-z}{b(z-\beta)} = \frac{1-\alpha^2}{b(\alpha-\beta)} = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

Cauchy's integral theorem,

$$\int_C \frac{1-z^2}{bz^2+2az+b}dz = 2\pi i \text{ [sum of residues of } f(z)]$$

$$= 2\pi i\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

$$(1) \Rightarrow I = R.P. of \frac{1}{i}2\pi i\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

$$= R.P. of \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

15. a) i) Find $L^{-1}\left(\frac{s^2}{\left(s^2+4\right)^2}\right)$ using convolution theorem.

Solution :

Let $F(s) = \frac{s}{s^2+4}$, $\quad G(s) = \frac{s}{s^2+4}$

$= \frac{s}{s^2+2^2}$, $\quad = \frac{s}{s^2+2^2}$

$f(t) = L^{-1}[F(s)]$ $\quad g(t) = L^{-1}[G(s)]$

$= L^{-1}\left[\frac{s}{s^2+2^2}\right]$ $\quad = L^{-1}\left[\frac{s}{s^2+2^2}\right]$

$= \cos 2t$ $\quad = \cos 2t$

$f(u) = \cos 2u$ $\quad g(t-u) = \cos 2(t-u)$

By Convolution theorem

$$L^{-1}[F(s).G(s)] = \int_0^t f(t).g(t-u)du \quad = \int_0^t \cos 2u \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos 2u\cos(2t-2u)\,du = \frac{1}{2}\int_0^t \left[\cos(2u+2t-2u)+\cos(2u-2t+2u)\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos 2t+\cos(4u-2t)\right]du = \frac{1}{2}\left[u\cos 2t - \frac{\sin(4u-2t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[\left(t\cos 2t+\frac{\sin 2t}{4}\right)-\left(0+\frac{\sin(0-2t)}{4}\right)\right] = \frac{1}{2}\left[t\cos 2t+\frac{\sin 2t}{4}+\frac{\sin 2t}{4}\right]$$

$$= \frac{1}{2}\left[t\cos 2t+\frac{\sin 2t}{2}\right]$$

15. a) ii) Find the Laplace transform of the Half wave rectifier $f(t)=\begin{cases}\sin wt, & 0<t<\pi/\omega\\ 0, & \pi/w<t<2\pi/w\end{cases}$ and

$f\left(t+2\pi/w\right)=f(t)$ for all t.

Solution : The given function is periodic with period $\frac{2\pi}{w}$,

By definition:

$$L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt \;=\frac{1}{1-e^{-2\pi s/w}}\int_0^{2\pi/w} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2\pi s/w}}\left[\int_0^{\pi/w} e^{-st}\sin wt dt+\int_{\pi/w}^{2\pi/w} e^{-st}(0)dt\right]$$

$$=\frac{1}{1-e^{-2\pi s/w}}\int_0^{\pi/w} e^{-st}\sin wt dt \;=\frac{1}{1-e^{-2\pi s/w}}\left[\frac{e^{-st}}{s^2+w^2}\left(-s\sin wt-w\cos wt\right)\right]_0^{\pi/w}$$

$$=\frac{-1}{\left(1-e^{-2\pi s/w}\right)\left(s^2+w^2\right)}\left[se^{-st}\sin wt+we^{-st}\cos wt\right]_0^{\pi/w}$$

$$=\frac{-w}{\left(1-e^{-2\pi s/w}\right)\left(s^2+w^2\right)}\left(e^{-s\pi/w}\cos\pi-e^0\cos^0\right)$$

$$=\frac{-w}{\left(1-e^{-2\pi s/w}\right)\left(s^2+w^2\right)}\left(-e^{-s\pi/w}-1\right)$$

$$=\frac{w\left(e^{-\pi s/w}+1\right)}{\left(1-e^{-2\pi s/w}\right)\left(s^2+w^2\right)}=\frac{w}{\left(s^2+w^2\right)\left(1-e^{-\pi s/w}\right)}$$

15. b) i) Find $L\left[\frac{\cos t-\cos bt}{t}\right]$

Solution : $L\left[\frac{\cos at-\cos bt}{t}\right]=\int_s^\infty\left\{L[\cos at]-L[\cos bt]\right\}ds$

$$=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds \;=\frac{1}{2}\int_s^\infty\left(\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right)ds$$

$$=\frac{1}{2}\left[\log\left(s^2+a^2\right)-\log\left(s^2+b^2\right)\right]_s^\infty \;=\frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2}{s^2}\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]_s^{\infty} = \frac{1}{2}\left[\log 1 - \log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

15. b) ii) Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution : Given $x'' - 3x' + 2x = 2$

$$L[x''] - 3L[x'] + 2L[x] = L[2]$$

$$[s^2L(x) - sx(0) - x'(0)] - 3[sL(x) - x(0)] + 2L(x) = 2 \times \frac{1}{s}$$

$$(s^2 - 3s + 2)L(x) - 5 = \frac{2}{s} \Rightarrow (s^2 - 3s + 2)L(x) = \frac{2}{s} + 5 = \frac{2+5s}{s}$$

$$\Rightarrow L(x) = \frac{5s+2}{s(s^2-3s+2)} = \frac{5s+2}{s(s-1)(s-2)}$$

Consider,

$$\frac{5s+2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$5s + 2 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

When

$s = 1,\ 7 = -B \Rightarrow B = -7$

$s = 2,\ 12 - 2C \Rightarrow C = 6$

$s = 0,\ 2 = 2A \Rightarrow A = 1$

$$\therefore L(x) = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$$x = L^{-1}\left(\frac{1}{s}\right) - 7L^{-1}\left(\frac{1}{s-1}\right) + 6L^{-1}\left(\frac{1}{s-2}\right)$$

$$= 1 - 7e^t + 6e^{2t}$$

MA 2161 – Mathematics – II
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Question Paper No. V 4012

Part – A

1. Solve the equation $\left(D^2-6D+13\right)y=0$.

Solution: Auxiliary equation is $m^2-6m+13=0$

Solving, $m=\dfrac{6\pm\sqrt{36-52}}{2}=\dfrac{6\pm4i}{2}=3\pm2i$

$\therefore$ Complementary function = $e^{3x}\,(A\cos2x+B\sin2x)$

$\therefore$ Solution is $y=e^{3x}\,(A\cos2x+B\sin2x)$

2. Find the particular integral of $\left(D+1\right)^2 y=e^{-x}\cos x$

Solution: $PI=\dfrac{1}{(D+1)^2}e^{-x}\cos x=e^{-x}\dfrac{1}{\left(D-1+1\right)^2}\cos x=e^{-x}\dfrac{1}{D^2}\cos x$

$=\dfrac{e^{-x}\cos x}{-1}=-e^{-x}\cos x$

3. Find grad $\left(r^n\right)$ where $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $r=\left|\vec{r}\right|$

Solution: grad$\left(r^n\right)=\nabla\left(r^n\right)=\sum\vec{i}\dfrac{\partial}{\partial x}\left(r^n\right)=\sum\vec{i}\,nr^{n-1}\dfrac{\partial r}{\partial x}$

$=\sum\vec{i}\;nr^{n-1}\dfrac{x}{r}=\sum\vec{i}\;nr^{n-2}x$

$=nr^{n-2}\sum\vec{i}x$

$=nr^{n-2}\;\vec{r}$

4. Find the unit normal to the surface $x^2+xy+z^2=4$ at $\left(1,-1,2\right)$.

Solution: $\phi=x^2+xy+z^2-4$

$\nabla\phi=\vec{i}\dfrac{\partial}{\partial x}\left(x^2+xy+z^2-4\right)+\vec{j}\dfrac{\partial}{\partial y}\left(x^2+xy+z^2-4\right)+\vec{k}\dfrac{\partial}{\partial z}\left(x^2+xy+z^2-4\right)$

$=\vec{i}\left(2x+y\right)+\vec{j}(x)+\vec{k}(2z)$

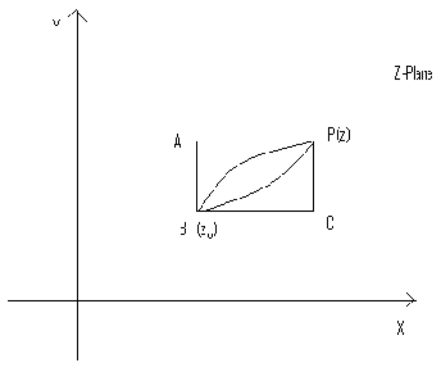
$\nabla\phi\left(1,-1,2\right)=\vec{i}+\vec{j}+4\vec{k}$

Unit vector normal to the surface $x^2+xy+z^2=4$ at $\left(1,-1,2\right)$ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{1+1+16}} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}$$

5. State the basic difference between the limit of a function of a real variable and that of a complex variable.

Solution: If $f(x)$ is a function of a real variable x then $\lim_{x\to a} f(x) = l$ implies $f(x)$ always approaches l when x approach a (either from left or from right) along x – axis (the line). If $f(z)$ is a function of complex variable z then $\lim_{z\to z_0} f(z) = b$ implies that in whatever way z approach z_0 (either along the path PAB or PCB or along the line PB or along the curve joining PB) $f(z)$ always approaches b.



6. Prove that a bilinear transformation has at most two fixed points.

Solution: A transformation of the form $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants is called bilinear transformation.

The point z is said to be fixed if the image of z under the transformation is z itself.

that is, $z = \frac{az+b}{cz+d} \Rightarrow cz^2 + dz = az + b \Rightarrow cz^2 + (d-a)z - b = 0$

Being a quadratic equation in z it has at most two roots and hence the bilinear transformation has at most two fixed points.

7. Identify the type of singularities of the following function: $f(z) = e^{\frac{1}{z-1}}$

Solution: The singularity of the function $f(z) = e^{\frac{1}{z-1}}$ is $z = 1$(isolated). The Laurent's series expansion of $f(z)$ is $1 + \frac{\left(\frac{1}{z-1}\right)^1}{\lfloor 1} + \frac{\left(\frac{1}{z-1}\right)^2}{\lfloor 2} + \ldots$ As the Laurent series expansion has infinitely many terms, $z = 1$ is an essential singularity.

8. Calculate the residue of $f(z)=\dfrac{e^{2z}}{(z+1)^2}$ at its pole.

Solution: $z=-1$ is a pole of order 2

$$\therefore \{\text{Res } f(z)\}_{z=-1}=\frac{1}{\lfloor 1}\lim_{z\to -1}\frac{d}{dz}(z+1)^2\frac{e^{2z}}{(z+1)^2}=\lim_{z\to -1}\frac{d}{dz}\left(e^{2z}\right)=\lim_{z\to -1}2e^{2z}=2e^{-2}$$

9. Find the Laplace transform of $t\cos at$

Solution:

$$L\{t\cos at\}=-\frac{d}{ds}L\{\cos at\}=\frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)=-\left[\frac{(s^2+a^2)-s(2s)}{(s^2+a^2)^2}\right]=-\left[\frac{-s^2+a^2}{(s^2+a^2)^2}\right]=\frac{s^2-a^2}{(s^2+a^2)^2}$$

10. Verify initial value theorem for $f(t)=1+e^{-t}(\sin t+\cos t)$.

Solution: Initial value theorem states that $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$

$$F(s)=L\{f(t)\}=L\{1+e^{-t}(\sin t+\cos t)\}=L\{1\}+L\{e^{-t}\sin t\}+L\{e^{-t}\cos t\}$$

$$=\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{(s+1)}{(s+1)^2+1}$$

$$\text{LHS}=\lim_{t\to 0} f(t)=\lim_{t\to 0} 1+e^{-t}(\sin t+\cos t)=2$$

$$\text{RHS}=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} s\left[\frac{1}{s}+\frac{1}{s^2+2s+2}+\frac{(s+1)}{s^2+2s+2}\right]$$

$$=\lim_{s\to\infty}\left[1+\frac{s}{s^2+2s+2}+\frac{s(s+1)}{s^2+2s+2}\right]=\lim_{s\to\infty}\left[1+\frac{\frac{s}{s^2}}{1+\frac{2}{s}+\frac{2}{s^2}}+\frac{1+\frac{1}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right]=2$$

Part – B

11. a) (i) Solve the equation $(D^2-3D+2)y=2\cos(2x+3)+2e^x$

Solution: General Solution $y=$Complementary Function (CF)+Particular Integral(PI)

Auxiliary equation is $m^2-3m+2=0\Rightarrow(m-2)(m-1)=0\Rightarrow m=2,1$

$\therefore \text{CF} = Ae^{2x} + Be^{x}$

$$PI = \frac{1}{D^2-3D+2}\,[2\cos(2x+3) + 2e^x\,]$$

$$= 2\left[\frac{1}{D^2-3D+2}\cos 2x\cos 3 - \frac{1}{D^2-3D+2}\sin 2x\sin 3 + \frac{1}{D^2-3D+2}e^x\right]$$

$$= 2\left[\cos 3\frac{1}{-4-3D+2}\cos 2x - \sin 3\frac{1}{-4-3D+2}\sin 2x + x\frac{1}{2D-3}e^x\right]$$

$$= -2\cos 3\frac{1}{3D+2}\cos 2x + 2\sin 3\frac{1}{3D+2}\sin 2x - xe^x$$

$$= -2\cos 3\frac{1}{(3D+2)}\frac{(3D-2)}{(3D-2)}\cos 2x + 2\sin 3\frac{1(3D-2)}{(3D+2)(3D-2)}\sin 2x - xe^x$$

$$= -2\cos 3\frac{1}{9D^2-4}(3D-2)\cos 2x + 2\sin 3\frac{1}{9D^2-4}(3D-2)\sin 2x - xe^x$$

$$= \frac{-2}{-40}\cos 3(3D-2)\cos 2x + \frac{2}{-40}\sin 3(3D-2)\sin 2x - xe^x$$

$$= \frac{1}{20}\cos 3(-6\sin 2x - 2\cos 2x) - \frac{1}{20}\sin 3(6\cos 2x - 2\sin 2x) - xe^x$$

a) ii) Apply the method of variation of parameters to solve $(D^2+4)y = \cot 2x$.

Solution: Auxiliary equation is $m^2 + 4 = 0$

i.e., $m^2 = -4 \Rightarrow m = \pm 2i$

$\therefore CF = A\cos 2x + B\sin 2x$

Let the solution be $y = A\cos 2x + B\sin 2x$, where arbitrary constants A and B are considered as suitable functions of x.

$$A = \int \frac{-\cot 2x\ \sin 2x}{W}\,dx + C_1$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$\therefore A = -\frac{1}{2}\int \frac{\cos 2x}{\sin 2x}\sin 2x\,dx + C_1 = -\frac{1}{2}\frac{\sin 2x}{2} + C_1 = -\frac{\sin 2x}{4} + C_1$$

$$B = \int \frac{\cot 2x\cos 2x}{w}dx + C_2$$

$$= \frac{1}{2}\int \frac{\cos 2x}{\sin 2x}\cos 2x\,dx + C_2 = \frac{1}{2}\int\left(\frac{1-\sin^2 2x}{\sin 2x}\right)dx + C_2$$

$$= \frac{1}{2}\int \operatorname{cosec} 2x\,dx - \frac{1}{2}\int \sin 2x dx + C_2 = \frac{1}{2}\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{1}{2}\frac{\cos 2x}{2} + C_2$$

$$\therefore y = \left(\frac{-\sin 2x}{4} + 4\right)\cos 2x + \left(\frac{\log\left(\cos ec2x - \cot 2x\right)}{4}\right) + \frac{\cos 2x}{4} + C_2)\sin 2x$$

$$y = C_1 \cos 2x + C2 \sin 2x + \frac{\sin 2x \log\left(\cos ec2x - \cot 2x\right)}{4}$$

11. b) i) Solve $\left(x^2 D^2 - xD + 4\right) y = x^2 \sin\left(\log x\right)$.

Solution: Put $Z = \log x$, $x = e^z$

$\therefore xDy = D'y \qquad D = \frac{d}{dx},\ D' = \frac{d}{dz}$

$x^2 D^2 y = D'\left(D'-1\right) y$

$\therefore x^2 D^2 y - xDy + 4y = x^2 \sin\left(\log x\right)$

reduces to $D'\left(D'-1\right) y - D'y + 4y = e^{2z} \sin z$

$\left(D'^2 - D' - D' + 4\right) y = e^{2z} \sin z$

A.E. is $m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4 - 4(4)}}{2} = \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm i\sqrt{3}$$

$$\therefore CF = e^z \left(A \cos\sqrt{3}z + B \sin\sqrt{3}z\right)$$

$$PI = \frac{1}{D'^2 - 2D' + 4} e^{2Z} \sin z$$

$$= e^{2z} \frac{1}{\left(D'+2\right)^2 - 2\left(D'+2\right) + 4} \sin z = e^{2z} \frac{1}{D'^2 + 4 + 4D' - 2D' - 4 + 4} \sin z$$

$$= e^{2z} \frac{1}{D'^2 + 2D' + 4} \sin z = e^{2z} \frac{1}{-1 + 2D' + 4} \sin z$$

$$= e^{2z} \frac{1}{2D'+3} \sin z = e^{2z} \frac{1}{\left(2D'+3\right)} \frac{\left(2D'-3\right)}{\left(2D'-3\right)} \sin z$$

$$= e^{2z} \frac{\left(2D'-3\right)}{4D'^2 - 9} \sin z = e^{2z} \frac{\left(2D'-3\right)}{4(-1) - 9} \sin z$$

$$= e^{2z} \frac{1}{13}\left(-2\cos z + 3\sin z\right)$$

$\therefore$ General solution is

$$y = e^{\log x}\left(A\cos\left(\sqrt{3}\log x\right) + B\sin\left(\sqrt{3}\log x\right)\right) - \frac{e^{2(\log x)}}{13}\left(2\cos(\log x) - 3\sin(\log x)\right)$$

$$= x\left(A\cos(\sqrt{3}\log x) + B\sin(\sqrt{3}\log x\right) - \frac{x^2}{13}\left(2\cos(\log x) - 3\sin(\log x)\right).$$

11. b) ii) Solve the simultaneous differential equations $\frac{dx}{dt}+2y=\sin 2t$, $\frac{dy}{dt}-2x=\cos 2t$.

Solution :

$Dx+2y=\sin 2t$ ----------> (1)

$Dy-2x=\cos 2t$ -------- --> (2)

multiplying eqn. (1) by 2 and operating eqn. (2) by D

$2Dx+4y=2\sin 2t$

$D^2y-2Dx=D(\cos 2t)=-2\sin 2t$

$\left(D^2+4\right)y=0$

A.E. is $m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\pm 2i$

$\therefore y=A_1\cos 2t+B_1\sin 2t -----> (3)$

Sub eqn. (3) in eqn. (2)

$D\left(A_1\cos 2t+B_1\sin 2t\right)-2x=\cos 2t$

$-2A_1\sin 2t+2B_1\cos 2t-2x=\cos 2t$

$2x=-2A_1\sin 2t+\left(2B_1-1\right)\cos 2t$

$\therefore x=-A_1\sin 2t+\frac{1}{2}\left(2B_1-1\right)\cos 2t$

$x=A_2\sin 2t+B_2\cos 2t$ where $A_2=-A_1$; $B_2=\frac{1}{2}\left(2B_1-1\right)$

12. a) i) Prove that curl $\left(\vec{u}\times\vec{v}\right)=\left(\vec{v}.\nabla\right)\vec{u}-\left(\vec{u}.\nabla\right)\vec{v}+\vec{u}\,div\,\vec{v}-\vec{v}\,div\,\vec{u}$.

Solution:

$$\nabla\times\left(\vec{u}\times\vec{v}\right)=\sum\vec{i}\times\frac{\partial}{\partial x}\left(\vec{u}\times\vec{v}\right)$$

$$=\sum\vec{i}\times\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}+\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$=\sum\vec{i}\times\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right)+\sum\vec{i}\times\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$=\left(\sum\vec{i}.\vec{v}\right)\frac{\partial\vec{u}}{\partial x}-\left(\sum\vec{i}.\frac{\partial\vec{u}}{\partial x}\right)\vec{v}+\left(\sum\vec{i}.\frac{\partial\vec{v}}{\partial x}\right)\vec{u}-\left(\sum\vec{i}.\vec{u}\right)\frac{\partial\vec{v}}{\partial x}$$

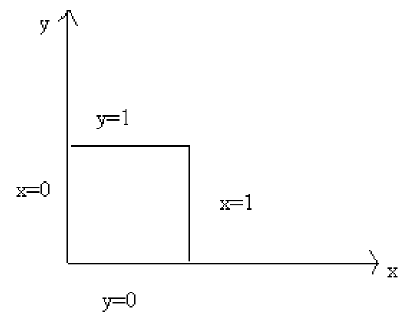
$$=\vec{v}.\left(\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{u}-\left(\nabla.\vec{u}\right)\vec{v}+\left(\nabla.\vec{v}\right)\vec{u}-\vec{u}.\left(\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{v}$$

$$=\left(\vec{v}.\nabla\right)\vec{u}-\left(\nabla.\vec{u}\right)\vec{v}+\left(\nabla.\vec{v}\right)\vec{u}-\left(\vec{u}.\nabla\right)\vec{v}$$

a) ii) Evaluate $\int_c \left(x^2+xy\right)dx+\left(x^2+y^2\right)dy$ where C is the square bounded by the lines $x=0,\ x=1,\ y=0$ and $y=1$

Solution:

By Green's theorem in the plane,



$$\int_c Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$P=x^2+xy,\ Q=x^2+y^2$$

$$\frac{\partial P}{\partial y}=x,\ \frac{\partial Q}{\partial x}=2x$$

$$\therefore \int_c \left(x^2+xy\right)dx+\left(x^2+y^2\right)dy=\int_{y=0}^{y=1}\int_{x=0}^{x=1}(2x-x)dxdy$$

$$=\int_{y=0}^{y=1}\left[\frac{x^2}{2}\right]_0^1 dy=\frac{1}{2}[y]_0^1=\frac{1}{2}$$

12. b) i) Verify Stoke's theorem when $\vec{F}=\left(2xy-x^2\right)\vec{i}-\left(x^2-y^2\right)\vec{j}$ and C is the boundary of the region enclosed by the parabolas $y^2=x$ and $x^2=y$.

Solution: By Stoke's theorem,

$$\oint_C \vec{F}.d\vec{r}=\iint_S\left(\nabla\times\vec{F}\right).\hat{n}dS$$

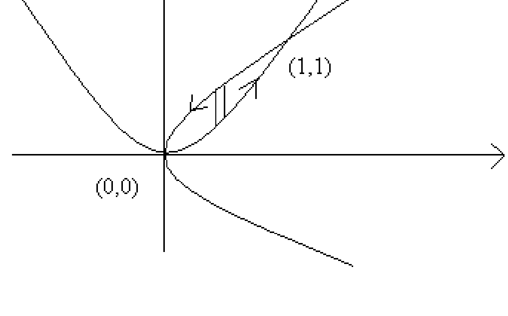
$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy-x^2 & -\left(x^2-y^2\right) & 0\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}\left(x^2-y^2\right)\right]-\vec{j}\left[\frac{\partial}{\partial x}(0)-\frac{\partial}{\partial z}\left(2xy-x^2\right)\right]+\vec{k}\left[\frac{\partial}{\partial x}\left(-x^2+y^2\right)-\frac{\partial}{\partial y}\left(2xy-x^2\right)\right]$$

$$\therefore \nabla\times\vec{F}=\vec{k}\left[-2x-2x\right]=-4x\vec{k}$$

$$RHS=\iint_s\left(\nabla\times\vec{F}\right).\hat{n}ds=\iint\left(-4x\vec{k}\right).\vec{k}\,dxdy$$

$$=-4\int_{x=0}^{x=1}x\left[\int_{y=x^2}^{y=\sqrt{x}}dy\right]dx$$



$$= -4\int_{x=0}^{x=1} x\left[y\right]_{x^2}^{\sqrt{x}} dx$$

$$= -4\int_{x=0}^{x=1} x\left[\sqrt{x} - x^2\right] dx$$

$$= -4\int_{x=0}^{x=1} \left(x^{\frac{3}{2}} - x^3\right) dx$$

$$= -4\left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^4}{4}\right]_0^1$$

$$= -4\left[\frac{2}{5} - \frac{1}{4}\right] = -4\left[\frac{8-5}{20}\right]$$

$$= -\frac{3}{5}$$

$$LHS = \int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r}$$

along C_1, $x^2 = y,\ dy = 2x\,dx$, x varies from 0 to 1

along C_2, $y^2 = x,\ dx = 2y\,dy$, y varies from 1 to 0

$$LHS = \int_c \left(2xy - x^2\right) dx - \left(x^2 - y^2\right) dy$$

$$= \int_{x=0}^{x=1} \left(2x(x^2) - x^2\right) dx - \left(x^2 - x^4\right) 2x\,dx \ + \int_{y=1}^{y=0} \left(2(y^2)y - y^4\right) 2y dy - \left(y^4 - y^2\right) dy$$

$$= \int_{x=0}^{x=1} \left(2x^3 - x^2 - 2x^3 + 2x^5\right) dx + \int_{y=1}^{y=0} \left(4y^4 - 2y^5 - y^4 + y^2\right) dy$$

$$= \left[\frac{-x^3}{3} + \frac{2x^6}{6}\right]_0^1 + \left[\frac{3y^5}{5} - \frac{2y^6}{6} + \frac{y^3}{3}\right]_1^0 = \left(\frac{-1}{3} + \frac{1}{3}\right) + \left(-\left(\frac{3}{5} - \frac{2}{6} + \frac{1}{3}\right)\right) = \frac{-3}{5}$$

12. b) ii) Evaluate $\int_c \left(\sin z dx - \cos x dy + \sin y dz\right)$ by using Stoke's theorem, where C is the boundary of the rectangle defined by $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

Solution: By Stoke's theorem,

$$\oint_C \vec{F}.d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right).\hat{n} dS$$

$$\vec{F} = \sin z\vec{i} - \cos x\vec{j} + \sin y\vec{x}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(\sin y) - \frac{\partial}{\partial z}(-\cos x)\right) - \vec{j}\left(\frac{\partial}{\partial x}(\sin y) - \frac{\partial}{\partial z}(\sin z)\right) + \vec{k}\left(\frac{\partial}{\partial x}(-\cos x) - \frac{\partial}{\partial y}(\sin z)\right)$$

$$= \cos y\vec{i} + \sin x\vec{k}$$

$$\hat{n} = \vec{k}$$

$$\therefore \iint_s \left(\nabla \times \vec{F}\right).\hat{n}\, ds = \int_{y=0}^{y=1} \int_{x=0}^{\pi} \left(\cos y\vec{i} + \sin x\vec{k}\right)\vec{k}\, dxdy$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=\pi} \sin x dx\, dy$$

$$= \int_{y=0}^{y=1} \left[-\cos x\right]_0^{\pi} dy$$

$$= 2\int_{y=0}^{y=1} dy = 2\left[y\right]_0^1 = 2$$

$$\therefore \int_c \left(\sin z dx - \cos x dy + \sin y dz\right) = 2$$

13. a) i) Verify that the families of curves $u = C_1$ and $v = C_2$ cut orthogonally when $u + iv = z^3$.

Solution : $u + iv = \left(x + iy\right)^3 = x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2$

$$\therefore u = x^3 - 3xy^2$$

$$u = -y^3 + 3x^2 y$$

since $u = C_1$, $x^3 - 3xy^2 = C_1$

Differentiate with respect to x,

$$3x^2 - 3\left(y^2 + x2y\frac{dy}{dx}\right) = 0$$

$$3x^2 - 3y^2 = 6xy\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1 \quad \text{(say)}$$

Since $u = C_2$, $-y^3 + 3x^2 y = C_2$

Difference with respect to x,

$$-3y^2\frac{dy}{dx}+3\left(2xy+x^2\frac{dy}{dx}\right)=0$$

$$6xy=\frac{dy}{dx}\left(3y^2-3x^2\right)$$

$$\therefore \frac{dy}{dx}=\frac{2xy}{y^2-x^2}=m_2 \text{ (say)}$$

$$\therefore m_1\, m_2=\left(\frac{x^2-y^2}{2xy}\right)\left(\frac{-2xy}{x^2-y^2}\right)=-1$$

$\therefore$ families of curves $u=C_1$ and $v=C_2$ cut orthogonally

13. a) ii) Find the analytic function $u+iv$, if $u=(x-y)\left(x^2+4xy+y^2\right)$. Also find the conjugate harmonic function v.

Solution :

$$\frac{\partial u}{\partial x}=\left(x^2+4xy+y^2\right)+(x-y)(2x+4y)$$

$$\frac{\partial u}{\partial y}=(-1)\left(x^2+4xy+y^2\right)+(x+y)(4x+2y)$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \quad \text{(by C – R equations)}$$

$$f'(z)=\frac{\partial u}{\partial x}(z,0)-i\frac{\partial u}{\partial y}(z,0)$$

$$=\left(z^2+2z^2\right)-i\left(-z^2+4z^2\right)$$

$$=3z^2-i\left(3z^2\right)$$

$$f'(z)=3(1-i)z^2$$

$$f(z)=3(1-i)\frac{z^3}{3}+C$$

$$f(z)=u+iv=(1-i)(x+iy)^3+(C_1+iC_2)$$

$$u+iv=(1-i)\left(x^3-iy^3+i^3x^2y-3xy^2\right)+(C_1+iC_2)$$

$$=\left(x^3-3xy^2-y^3+3x^2y\right)+C_1+i\left(3x^2y-y^3-x^3+3xy^2+C_2\right)$$

$$\therefore v=3x^2y-y^3-x^3+3xy^2+C_2$$

$$=3xy(x+y)-(x^3+y^3)+C_2$$

$$=3xy(x+y)-(x+y)(x^2-xy+y^2)+C_2$$

$$= (x+y)(4xy - x^2 - y^2) + C_2$$

13. b) i) Find the image of the circle $|z-1| = 1$ in the complex plane under the mapping $w = \dfrac{1}{z}$.

Solution :

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$|z-1| = 1 \;\Rightarrow \left|\frac{1}{w} - 1\right|$$

$$\frac{|1-w|}{|w|} = 1 \quad \Rightarrow |1-w| = |w|$$

$$\therefore |1-w|^2 = |w|^2 \quad (w = u + iv)$$

$$|(1-u) - iv|^2 = |u+iv|^2$$

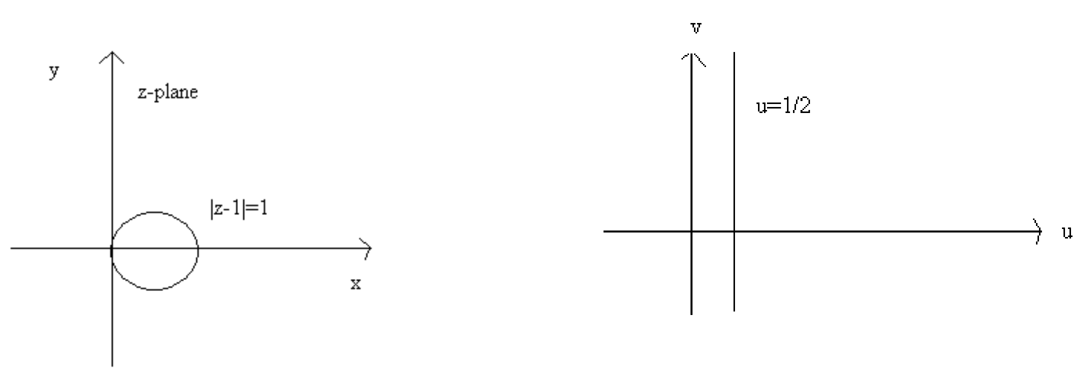
$$\therefore (1-u)^2 + (-v)^2 = u^2 + v^2$$

$$u^2 + 1 - 2u + v^2 = u^2 + v^2$$

$$\therefore 1 - 2u = 0$$

$$u = \frac{1}{2}$$

$\therefore$ image of the circle $|z-1| = 1$ under $w = \dfrac{1}{z}$ is $u = \dfrac{1}{2}$, which is a straight line in the w - plane.



13. b) ii) When the function $f(z) = u + iv$ is analytic . Prove that the curves u = constant and v = constant are orthogonal.

Solution :

Since $u = C_1,\; du = 0$

$$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \qquad \left(\because du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)$$

$$\therefore \frac{\partial u}{\partial y}dy = -\frac{\partial u}{\partial x}dx$$

$$m_1 = \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \rightarrow (1)$$

Since $v = C_2, \quad dv = 0$

$$= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

By C – R equations , $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\therefore m_2 = \frac{\partial y}{\partial x} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} \rightarrow (2)$$

$$\therefore m_1 m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1$$

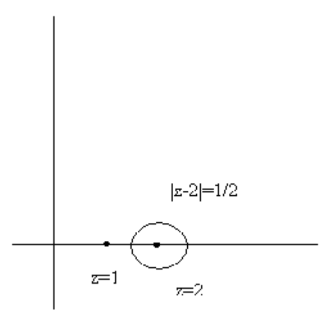
$\therefore$ curves u =constant and v = constant are orthogonal.

14. a) i) Evaluate $\int_C \frac{zdz}{(z-1)(z-2)^2}$ where C is $|z-2| = \frac{1}{2}$ by using Cauchy's integral formula.

Solution :

The function $\frac{z}{(z-1)(z-2)^2}$ has singularities z = 1, 2 of which z = 2 lies within C.

$$\therefore \int_C \frac{zdz}{(z-1)(z-2)^2} = \int_C \frac{\left(\frac{z}{z-1}\right)dz}{(z-2)^2}$$



By Cauchy's integral formula for derivatives,

$$\int_C \frac{f(z)dz}{(z-a)^2} = \frac{2\pi i}{\lfloor 1} f'(a)$$

$$\therefore \int_C \frac{\left(\frac{z}{z-1}\right)dz}{(z-2)^2} = \frac{2\pi i}{\lfloor 1} f'(2)$$

Where $f(z) = \frac{z}{z-1}$, $f'(z) = \frac{(z-1)-z}{(z-1)^2}$, $= \frac{-1}{(z-1)^2}$

$$\therefore f'(z) = \frac{-1}{(z-1)^2} = -1$$

$$\therefore \int_C \frac{\left(\frac{z}{z-1}\right)dz}{(z-2)^2} = -2\pi i$$

14. a) ii) Evaluate $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for the regions $|z| > 3$ and $1 < |z| < 3$

Solution : $f(z) = \frac{1}{(z+1)(z+3)} = \frac{A}{(z+1)} + \frac{B}{(z+3)}$

$$\therefore 1 = A(z+3) + B(z+1)$$

$$z = -3 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$z = -1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$\therefore f(z) = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}$$

In the region $|z| > 3$, $|z| > 1$

$$\therefore \left|\frac{3}{z}\right| < 1 \,\&\, \left|\frac{1}{z}\right| < 1$$

$$\therefore f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+........\infty\right) - \frac{1}{2z}\left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+........\infty\right)$$

In the region $1 < |z| < 3$,

$$\left|\frac{1}{z}\right| < 1 \ \& \ \left|\frac{z}{3}\right| < 1$$

$$\therefore f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\,\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+........\infty\right) - \frac{1}{6}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+........\infty\right)$$

14. b) i) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Solution : Let $z = e^{i\theta}, \ |z| = 1, \ \ dz = ie^{i\theta}\, d\theta$

$$\therefore d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}, \ \ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{z} = \frac{z+1/2}{2}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{|z|=1} \frac{\frac{dz}{iz}}{z+\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_{|z|=1} \frac{\frac{dz}{iz}}{\frac{z^2+4z+1}{2z}} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+4z+1}$$

Singularities of $f(z) = \frac{1}{z^2+4z+1}$ are $-2+\sqrt{3}$, $-2\pm\sqrt{3}$

$$\left[z^2+4z+1=0, z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3} \right]$$

of the singularities, $-2+\sqrt{3}$ lies within C. $\therefore$ By Cauchy's Residue theorem

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\times 2\pi i \ \ (\text{Res } f(z) \text{ at } z=-2+\sqrt{3})$$

Res $f(z)$ at $z = -2+\sqrt{3}$

$$= lt_{z\to -2+\sqrt{3}}\left(z-(-2+\sqrt{3})\right)\frac{1}{\left(z-\left(-2+\sqrt{3}\right)\right)\left(z-(-2-\sqrt{3})\right)}$$

$$= \frac{1}{\left(-2+\sqrt{3}\right)-\left(-2-\sqrt{3}\right)} = \frac{1}{2\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = 4\pi \times \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

14. b) ii) Evaluate $\int_0^{\infty} \frac{dx}{\left(x^2+a^2\right)^3}$, $a > 0$ using contour integration.

Solution : We know that

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx$$

If $f(-x) = f(x)$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^3} = 2\int_0^{\infty} \frac{dx}{\left(x^2+a^2\right)^3}$$

$$\therefore \int_0^{\infty} \frac{dx}{\left(x^2+a^2\right)^3} = 2\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^3}$$

Consider the integral $\int_C f(z)dz$

Where $f(z) = \frac{1}{\left(z^2+a^2\right)^3}$

C is a closed curve consisting of a semicircle C_1 with centre at origin and radius R large enough such that all the singularities of $f(z)$ above the real axis are within C_1 and the line segment joining $-R$ to R along the real axis.

$$\therefore \int_C f(z)dz = \int_C f(z)dz + \int_{-R}^{R} f(x)dx$$

(As long real axis, $z = x$)

But $\int_C f(z)dz = 2\pi i$ (sum of residues of $f(z)$ at isolated singularities within C).

Singularities of $f(z) = \frac{1}{\left(z^2+a^2\right)^3}$ are $z = ai, -ai$ (both poles of order 3) of which $z = ai$ lies within C.

$$\therefore \left\{\text{Re}s\, f(z)\right\}_{z=ai} = \frac{1}{\underline{|2}} \lim_{z\to ai} \frac{d^2}{dz^2}\left(z-ai\right)^3 \frac{1}{\left(z-ai\right)^3\left(z+ai\right)^3}$$

$$= \frac{1}{2} \lim_{z\to ai} \frac{d}{dz}(-3)\left(z+ai\right)^{-4}$$

$$= \frac{1}{2}\lim_{z\to ai}(-3)(-4)(z+ai)^{-5}$$

$$= \frac{12}{2}\, lim \frac{1}{(z+ai)^5}$$

$$= 6\frac{1}{(2ai)^5} = 6\times\frac{1}{32a^5}\times\frac{1}{i}$$

$$\therefore \int_C f(z)dz = 2\pi i\times\frac{3}{16a^5 i} = \frac{3\pi}{8a^5}$$

$$\therefore \frac{3\pi}{8a^5} = \int_{C_1} f(z)dz + \int_{-R}^{R} f(x)dx$$

$As\, R \to \infty$

$on\, C_1\; z = \text{Re}^{i\theta},\; 0 \le \theta \le \pi$

$|z| = R$

$dz = i\,\text{Re}^{i\theta}\, d\theta$

$|dz| = Rd\theta$

$$\left|\int_{C_1} f(z)dz\right| \le \int_{C_1} |f(z)|\,|dz|$$

$$\le \int_{C_1} \frac{1}{|z^2+a^2|^3}\;|dz|$$

$$\le \int_{C1} \frac{1}{\left(|z|^2-a^2\right)^3}|dz|$$

$$\le \int_{\theta=0}^{\pi} \frac{1}{\left(R^2-a^2\right)^3} R\, d\theta$$

$$= \frac{R}{\left(R^2-a^2\right)^3}[\theta]_0^{\pi} = \frac{\pi R}{\left(R^2-a^2\right)^3}$$

$$\lim_{R\to\infty}\left|\int_{C_1} f(z)dz\right| \le \lim_{R\to\infty}\frac{\frac{\pi R}{R^6}}{\left(\frac{R^2-a^2}{R^2}\right)^3}$$

$$\lim_{R\to\infty}\frac{\frac{\pi R}{R^5}}{\left(1-\frac{a^2}{R^2}\right)^3} = 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^3} = \frac{3\pi}{8a^5} \text{ and } \int_{0}^{\infty} \frac{dx}{\left(x^2+a^2\right)^3} = \frac{1}{2} \times \frac{3\pi}{8a^5} = \frac{3\pi}{16a^5}$$

15. a) i) Using convolution theorem find the inverse Laplace transform of $\frac{1}{\left(s^2+1\right)\left(s+1\right)}$

Solution : By convolution theorem,

$$L^{-1}\{F(s)F(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\}$$

$$\therefore L^{-1}\left\{\frac{1}{\left(s^2+1\right)\left(s+1\right)}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} * L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \sin t * e^{-t} = \int_{u=0}^{t} (\sin u) e^{-(t-u)} du = e^{-t} \int_{0}^{t} e^{u} \sin u du$$

$$= e^{-t}\left[\frac{e^u}{1^2+1^2}\left(\sin u - \cos u\right)\right]_0^t = e^{-t}\left[\frac{e^t}{2}\left(\sin t - \cos t\right) = \left(\frac{1}{2}(-1)\right)\right]$$

$$= e^{-t}\left[\frac{e^t \sin t}{2} - \frac{e^t \cos t}{2} + \frac{1}{2}\right] = \frac{\sin t}{2} - \frac{\cos t}{2} + \frac{e^{-t}}{2}$$

15. a) ii) Find Laplace transform of $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \quad f\left(t + 2a\right) = f(t)$

Solution :

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} f(t) e^{-st} dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a t e^{-st} dt + \int_a^{2a} \left(2a - t\right) e^{-st} dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left((t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right)_0^a + \left((2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right)_a^{2a}\right]$$

$$= \frac{1}{1-e-^{2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1 + \left(e^{-as}\right)2 - 2e^{-as}}{s^2}\right] = \frac{1}{\left(1-e^{-as}\right)\left(1+e^{-as}\right)} \frac{\left(1-e^{-as}\right)^2}{s^2}$$

$$= \frac{1}{s^2}\frac{1-e^{-as}}{1+e^{-as}} = \frac{1}{s^2}\frac{1-\frac{e^{\frac{-as}{2}}}{e^{\frac{as}{2}}}}{1+\frac{e^{\frac{-as}{2}}}{e^{\frac{as}{2}}}} \qquad L\{f(t)\} = \frac{1}{s^2}\frac{e^{\frac{as}{2}}-e^{\frac{-as}{2}}}{e^{\frac{as}{2}}+e^{\frac{-as}{2}}} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

15. b) i) Find the Laplace transform of square wave function defined by $f(t) = \begin{cases} 1, & 0 < t < a \\ -1, & a < t < 2a \end{cases}$ with period 2a.

Solution :

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left[\int_0^{2a} f(t)e^{-st}dt\right] \qquad = \frac{1}{1-e^{-2as}}\left[\int_0^{a} 1e^{-st}dt + \int_a^{2a} -1e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[-\frac{e^{-as}}{3}+\frac{1}{3}+\frac{e^{-2as}}{s}-\frac{e^{-as}}{s}\right] = \frac{1}{1-e^{-2as}}\frac{\left(1+(e^{-as})^2-2e^{-as}\right)}{s}$$

$$= \frac{1}{\left(1-e^{-as}\right)\left(1+e^{-as}\right)}\frac{\left(1-e^{-as}\right)^t}{s}$$

$$= \frac{1}{s}\frac{1-e^{-as}}{1+e^{-as}} = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

15. b) ii) Solve the differential equation $\frac{d^2y}{dt^2} + y = \sin 2t$, $y(0) = 0$, $y'(0) = 0$ by using Laplace transform method.

Solution :

$$y'' + y = \sin 2t$$

Take Laplace transform on both sides,

$$L\{y''+y\} = L\{\sin 2t\}$$

$$L\{y''\} + L\{y\} = \frac{2}{s^2+4}$$

$$\left(s^2L\{y\} - sy(0) - y'(0)\right) + L\{y\} = \frac{2}{s^2+4}$$

$$\therefore \left(s^2+1\right)L\{y\} = \frac{2}{s^2+4} \qquad \therefore L\{y\} = \frac{2}{\left(s^2+1\right)\left(s^2+4\right)}$$

$$\therefore y = L^{-1}\left\{\frac{1}{s^2+1}\frac{2}{s^2+4}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2+1}\right\} * L^{-1}\left\{\frac{2}{s^2+4}\right\}$$ (by Convolution theorem)

$$= \sin t * \sin 2t$$

$$= \int_{u=0}^{t} \sin u \sin 2(t-u)\,du \quad = \frac{1}{2}\int_0^t \left[\cos\left(u-(2t-2u)\right) - \cos\left(u+(2t-2u)\right)\right] dy$$

$$= \frac{1}{2}\int_0^t \left[\cos(3u-2t) - \cos(2t-u)\right] du \quad = \frac{1}{2}\left[\frac{\sin(3u-2t)}{3} - \frac{\sin(2t-u)}{-1}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin t}{3} + \sin t - \frac{\sin(-2t)}{3} - \sin 2t\right]$$

$$= \frac{1}{2}\left[\frac{4}{3}\sin t - \frac{2}{3}\sin 2t\right] = \qquad y = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$$

MA 2161 – Mathematics – II
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Question Paper No. 53294

Part – A

1. Reduce the equation $\left(x^2D^2+xD+1\right)y=\log x$ into an ordinary differential equation with constant coefficients

Solution:

$Z=\log x\,,x=e^z \qquad D=\dfrac{d}{dx},D^1=\dfrac{d}{dz}$

$x^2D^2\ y=D^1\left(D^1-1\right)y \quad xDy=D^1y$

$\therefore\left(x^2D^2+xD+1\right)y=\log x$

becomes $\left(D^1\left(D^1-1\right)+D^1+1\right)y=z$

$\left(D^{1^2}-D^1+D^1+1\right)y=z$

Ie., $\left(D^{1^2}+1\right)y=z$.

2. Find the particular integral of $\left(D^2-2D+2\right)y=e^x\cos x$.

Solution:

$PI=\dfrac{1}{D^2-2D+2}e^x\cos x=e^x\dfrac{1}{\left(D+1\right)^2-2\left(D+1\right)+2}\cos x$

$=e^x\dfrac{1}{D^2+1+2D-2D-2+2}\cos x$

$=e^x\dfrac{1}{D^2+1}\cos x=e^x x\dfrac{1}{2D}\cos x$

$=\dfrac{xe^x}{2}\dfrac{1}{D}\cos x=\dfrac{xe^x}{2}\sin x$

3. Prove that div $\vec{r}=3$ and cure $\vec{r}=\vec{0}$

Solution:

$\vec{r}=x\vec{i}+y\vec{i}+z\vec{k}$

$$\nabla\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=\vec{i}\left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right]-\vec{j}\left[\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial z}(x)\right]+\vec{n}\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]$$

$=0\vec{i}-0\vec{j}+0\vec{k}=\vec{0}$

4. State Stoke's theorem.
   Solution:

If $\vec{F}$ is a differentiable vector point function defined on the open surface S, then

$$\int_C \vec{F}.d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right).\hat{n}\,ds$$

Where C is the curve bounding the open surface. $\vec{n}$ is a unit vector drawn normal to the surface.

5. Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

Solution:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \;;\; \frac{\partial^2 u}{\partial x^2} = 6x + 6$$

$$\frac{\partial u}{\partial y} = -6xy - 6y \quad ; \; \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0$$

$\therefore$ u is harmonic.

6. Verify whether $f(z) = \bar{z}$ analytic or not.

Solution:

$$f(z) = u + iv = \bar{z} = x - iy \;\; \therefore u = x, v = -y$$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Since the C-R equation $u_x = v_y$ is not satisfied $f(z) = \bar{z}$ is not analytic.

7. Evaluate $\oint_C \frac{e^z}{z-1}dz$ if C is $|z| = 2$.

Solution:

The function $\frac{e^z}{z-1}$ has singularity at $z = 1$ which lies inside C, $\therefore$ by Cauchy's integral formula

$$\oint_C \frac{e^z dz}{z-1} = 2\pi i f(1)$$

Where $f(z) = e^z$ and hence $f(1) = e$

$$\therefore \oint_C \frac{e^z dz}{z-1} = 2\pi i e$$

8. If $f(z)=\frac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+.......\right]$,

find the residue of $f(z)$ at $z=1$

Solution:

Residue of $f(z)$ at $z=1$ is the coefficient of $\frac{1}{z-1}$ in the Laurent's series expansion

i.e., $\left[\operatorname{Re}s\, f(z)\right]_{z=1}=-1$.

9. Find Laplace transform of $\mathrm{t}\sin 2\mathrm{t}$

Solution:

$$\mathrm{L}\{\mathrm{t}\sin 2\mathrm{t}\}=-\frac{\mathrm{d}}{\mathrm{ds}}\mathrm{L}\{\sin 2\mathrm{t}\}$$

since $L\{t\,f(t)\}=-\frac{d}{ds}L\{f(t)\}$

$$\therefore L\{t\sin 2t\}=-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$=-z(-1)\left(s^2+4\right)^{-2}(2s)$$

$$=\frac{4s}{\left(s^2+4\right)^2}$$

10. Find $L^{-1}\left[\frac{1}{s^2+4s+4}\right]$

Solution:

$$L^{-1}\left\{\frac{1}{s^2+4s+4}\right\}=L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$=e^{-2t}\,L^{-1}\left\{\frac{1}{s^2}\right\}=e^{-2t}$$

Since $L^{-1}\{F(s+a)\}=e^{-at}L^{-1}\{F(s)\}$

$$L^{-1}\left\{\frac{1}{s^2+4s+4}\right\}=e^{-2t}t$$

Part – B

11. a) i) Solve $\left(D^2+16\right)y=\cos^3 x$

Solution:

y =complementary function $(\mathrm{CF})+$ Particular integral (PI)

CF is the solution of $\left(\mathrm{D}^2+16\right)\mathrm{y}=0$

Auxiliary equation is $\mathrm{m}^2+16=0$

$$m^2=-16\Rightarrow m=\pm 4i$$

$\therefore CF = A\cos 4x + B\sin 4x$

$$PI=\frac{1}{D^2+16}\cos^3 x=\frac{1}{D^2+16}\left(\frac{3}{4}\cos x+\frac{1}{4}\cos 3x\right)$$

$$=\frac{3}{4}\ \frac{1}{D^2+16}\cos x+\frac{1}{4}\ \frac{1}{D^2+16}\cos 3x \quad =\frac{3}{4}\ \frac{1}{-1+16}\cos x+\frac{1}{4}\ \frac{1}{-9+16}\cos 3x$$

$$=\frac{3}{60}\cos x+\frac{1}{28}\cos 3x \qquad =\frac{1}{20}\cos x+\frac{1}{28}\cos 3x$$

$\therefore$ General solution

$$y=A\cos 4x+B\sin 4x+\frac{1}{20}\cos x+\frac{1}{28}\cos 3x$$

11. a) ii) Solve by the method of variation of parameters, $\frac{d^2y}{dx^2}+4y=\sec 2x$

Solution:

AE is $m^2+4=0$

$\therefore m=\pm 2i$

$\therefore CF = A\cos 2x + B\sin 2x$

Let the general solution be $y=A\cos 2x+B\sin 2x$

where A and B are considered to be functions of x.

$$A=\int\frac{-\sec 2x \times \sin 2x}{W}dx$$

$$W=\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}$$

$$=2\cos^2 2x+2\sin^2 2x=2\neq 0.$$

$$B=\int\frac{\sec 2x \times \cos 2x}{W}\ dx$$

$$A=\frac{-1}{2}\int\frac{1}{\cos 2x}\times\sin 2x\ dx \quad =\frac{1}{4}\int\frac{-2\sin 2x\ dx}{\cos 2x} \quad =\frac{1}{4}\int\frac{d(\cos 2x)}{\cos 2x}$$

$$=\frac{1}{4}\log(\cos 2x)+C_1$$

$$B=\frac{1}{2}\int\frac{1}{\cos 2x}\times\cos 2x\ dx=\frac{1}{2}\int dx=\frac{1}{2}x+C_2$$

$$\therefore\ y=\left(\frac{1}{4}\log\left(\cos 2x\right)+C_1\right)\cos 2x+\left(\frac{x}{2}+C_2\right)\sin 2x$$

$$y=\frac{1}{4}\cos 2x\log\left(\cos 2x\right)+\frac{x}{2}\sin 2x+C_2\sin 2x+C_1\cos 2x$$

11. b) i) Solve: $\left(x^2D^2-3xD+4\right)y=x^2\cos\left(\log x\right)$

Solution:

Let $z=\log x, x=e^z$

$$x^2D^2y=D^1\left(D^1-1\right)y,\ xDy=D^1y\quad\text{where } D=\frac{d}{dx}, D^1=\frac{d}{dz}$$

$\therefore\left(x^2D^2-3xD+4\right)y=x^2\cos\left(\log x\right)$ becomes

$$\left(D^1\left(D^1-1\right)-3D^1+4\right)y=e^{2z}\cos z$$

$$\left(D^{1^2}-4D^1+4\right)y=e^{2z}\cos z$$

AE is $m^2-4m+4=0$

$(m-2)^2=0\ \Rightarrow m=2,2$

$$\therefore CF=\left(A+Bz\right)e^{2z}=\left(A+B\log x\right)e^{2\log x}=\left(A+B\log x\right)x^2$$

$$PI=\frac{1}{D^{1^2}-4D^1+4}e^{2z}\cos z$$

$$=\frac{1}{\left(D^1-2\right)^2}e^{2z}\cos z=e^{2z}\frac{1}{\left(D^1+2-2\right)^2}\cos z$$

$$=e^{2z}\frac{1}{D^{1^2}}\cos z=e^{2z}\frac{1}{D^1}\sin z=-e^{2z}\cos z$$

$$=-e^{2\log x}\cos\left(\log x\right)=-x^2\cos\left(\log x\right)$$

$\therefore$ General solution is $y=\left(A+B\log x\right)x^2-x^2\cos\left(\log x\right)$

11. b) ii) Solve $\frac{dx}{dt}+2y=-\sin t, \frac{dy}{dt}-2x=\cos t$ given $x=1$ and $y=0$ at $t=0$

Solution:

$$Dx+2y=-\sin t-----(1)$$
$$Dy-2x=\cos t------(2)$$
$$D=\frac{d}{dt}$$

Multiply (1) by 2 and operate (2) by D

$$2Dx+4y=-2\sin t$$
$$-2Dx+D^2y=D(\cos t)=-\sin t$$
$$\overline{(D^2+4)y=-3\sin t}$$

Solving for y,

$$\text{A.E is } m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\pm 2i$$

$$\therefore \text{C.F.}=A\cos 2t+B\sin 2t$$

$$PI=\frac{1}{D^2+4}(-3\sin t)=-3\frac{1}{-1+4}\sin t=-\sin t$$

$$\therefore\ y = A\cos 2t+\sin 2t-\sin t------(3)$$

Substituting (3) in (2). We get

$$D(A\cos 2t+B\sin 2t-\sin t)-2x=\cos t$$
$$-2A\sin 2t+2B\cos 2t-\cos t-2x=\cos t$$
$$2x=-2A\sin 2t+2B\cos 2t-2\cos t$$

$$x=-A\sin 2t+B\cos 2t-\cos t------(4)$$

Given $x=1, y=0$ at $t=0$

$\therefore 0=A$; $1=B-1 \quad \Rightarrow\ B=2$

$\therefore\ x=2\cos 2t-\cos t, y=-\sin t+2\sin 2t$

12. a) i) If $\vec{r}$ is the position vector of the point (x,y,z) prove that $\nabla^2(r^n)=n(n+1)\ r^{n-2}$

Solution:

$$\nabla^2(r^n)=\nabla.\nabla(r^n)$$

Now $\nabla(r^n)=\sum\vec{i}\frac{\partial}{\partial x}(r^n)=\sum\vec{i}\ nr^{n-1}\frac{x}{r}=nr^{n-2}\sum\vec{i}\ x$

$$= n\, r^{n-2}\ \vec{r}$$

$$\therefore \nabla^2\left(r^n\right) = \nabla.\left(n r^{n-2}\ \vec{r}\right)$$

$$\nabla\,.\left(\phi\vec{F}\right) = \phi\left(\nabla.\vec{F}\right) + \nabla\phi.\vec{F}$$

$$\therefore \nabla^2\left(r^n\right) = n r^{n-2}\left(\nabla.\vec{r}\right) + \nabla\left(n r^{n-2}\right).\vec{r}$$

$$\nabla.\vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\nabla\left(n r^{n-2}\right) = \sum \vec{i}\frac{\partial}{\partial x}\left(n r^{n-2}\right) = \sum \vec{i}\ n(n-2)r^{n-3}\frac{x}{r}$$

$$= \sum \vec{i}\ n(n-2)r^{n-4}x = n(n-2)r^{n-4}\sum \vec{i}\,x$$

$$= n(n-2)r^{n-4}\ \vec{r}$$

$$\therefore \nabla^2\left(r^n\right) = 3n r^{n-2} + n(n-2)r^{n-4}\ \vec{r}.\vec{r}$$

$$= 3n r^{n-2} + n(n-2)r^{n-4}\ r^2$$

$$= 3n r^{n-2} + n(n-2)r^{n-2}$$

$$= \left(n^2 - 2n + 3n\right)r^{n-2} = \left(n^2 + n\right)r^{n-2}$$

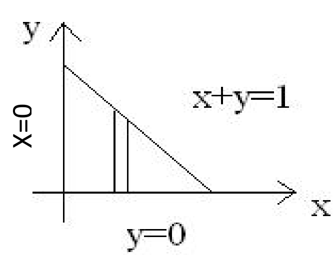
$$= n(n+1)r^{n-2}$$

12. a) ii) Verify Green's theorem in plane for $\int_C \left[\left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy\right]$ where C is the boundary of the region bounded by the lines $x = 0, y = 0,$ and $x + y = 1$.

Solution:

By Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$



$$P = 3x^2 - 8y^2\,;\, Q = 4y - 6xy$$

$$\frac{\partial P}{\partial y} = -16y \quad ; \frac{\partial Q}{\partial x} = -6y$$

$$\therefore \text{RHS} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left(-6y - (-16y)\right)dy\,dx$$

$$=10\int_{x=0}^{x=1}\left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx = 5\int_{x=0}^{x=1}(1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_{x=0}^{x=1} = \frac{5}{3}$$

$$\text{LHS} = \int_C Pdx + dQy = \int_{\substack{y=0\\ dy=0\\ x=0 \text{ to } x=1}} Pdx + Qdy + \int_{\substack{x+y=1\\ y=1-x\\ dy=-dx\\ x=1 \text{ to } x=0}} Pdx + Qdy + \int_{\substack{x=0\\ dx=0\\ y=1 \text{ to } y=o}} Pdx + Qdy$$

$$\int_C Pdx + Qdy = \int_{x=0}^{x=1} 3x^2dx + \int_{x=1}^{x=0}\left(3x^2 - 8\left(1-x^2\right)\right)dx + \left(4(1-x) - 6x(1-x)(-dx)\right) + y = \int_{y=1}^{x=1} 4ydy$$

$$=\left[\frac{3x^3}{3}\right]_{x=0}^{x=1} + \int_{x=1}^{0}\left[\left(3x^2 - 8x^2 - 6x^2\right) + (16x + 4x + 6x) + (-8-4)\right]dx + \left[\frac{4y^2}{2}\right]_{y=1}^{y=0}$$

$$=1+\left[\frac{-11x^3}{3} + \frac{26x^2}{2} - 12x\right]_1^0 - 2 = -1 + \left[\frac{11}{3} - 13 + 12\right] = \frac{11}{3} - 2 = \frac{5}{3}$$

$\therefore$ LHS = RHS

12. b) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by

$x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

Solution:

By Gauss divergence theorem

$$\oiiint_V \nabla.\vec{F}\,dv = \oiint_S \vec{F}.\hat{n}\,ds$$

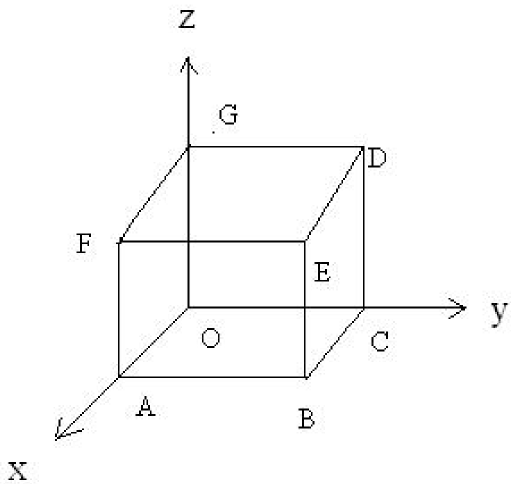
$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$=\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}\left(-y^2\right) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$$

$$\text{LHS} = \oiiint_V \nabla.\vec{F}dv = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1}(4z - y)\,dz\,dy\,dx$$

$$=\int_{x=0}^{x=1}\int_{y=0}^{y=1}\left[\frac{4z^2}{2} - yz\right]_0^1 dydx = \int_{x=0}^{x=1}\int_{y=0}^{y=1}(2-y)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left[2y - \frac{y^2}{2} \right]_0^1 dx = \frac{3}{2} \int_{x=0}^{x=1} dx = \frac{3}{2} [x]_0^1 = \frac{3}{2}$$



on $S_1(OABC), z = 0, \hat{n} = -\vec{k}, ds_1 = dxdy$

on $S_2(EDGF), z = 1, \hat{n} = \vec{k}, ds_2 = dxdy$

on $S_3(OAFG), Y = 0, \hat{n} = -\vec{j}, ds_3 = dxdz$

on $S_4(CBED), Y = 1, \hat{n} = \vec{j}, ds_4 = dxdz$

on $S_5(OCDG), x = 0, \hat{n} = -\vec{i}, ds_5 = dydz$

on $S_6(ABEF), x = 1, \hat{n} = \vec{i}, ds_6 = dydz$

$$RHS = \iint_S \vec{F}.\hat{n}\, ds = \iint_{S_1} \vec{F}.\hat{n}\, ds_1 + \iint_{S_2} \vec{F}.\hat{n}\, ds_2 + \iint_{S_3} \vec{F}.\hat{n}\, ds_3 + \iint_{S_4} \vec{F}.\hat{n}\, ds_4 + \iint_{S_5} \vec{F}.\hat{n}\, ds_5 + \iint_{S_6} \vec{F}.\hat{n}\, ds_6$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=1} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{k}\right) dx\, dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} -yz\, dx\, dy = 0 \quad (\text{since } z = 0 \text{ on } S_1)$$

$$\iint_{S_2} \vec{F}.\hat{n}\, ds_2 = \int_{y=0}^{y=1} \int_{x=0}^{x=1} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{k}\right) dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=1} y\, dx\, dy \quad (\text{since } z = 1 \text{ on } S_2)$$

$$= \int_{x=0}^{x=1} \left[\frac{y^2}{2} \right]_0^1 dx = \frac{1}{2} [x]_0^1 = \frac{1}{2}$$

$$\iint_{S_3} \vec{F}.\hat{n}\, ds_3 = \int_{z=0}^{z=1} \int_{x=0}^{x=1} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{j}\right) dx\, dz = \int_{z=0}^{z=1} \int_{x=0}^{x=1} \left(+y^2\right) dx\, dz = 0 \quad (\because y = 0 \text{ on } S_3)$$

$$\iint_{S4} \vec{F}.\hat{n}\, ds_4 = \int_{z=0}^{z=1} \int_{x=0}^{x=1} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{j}\right) dx\, dz = \int_{z=0}^{z=1} \int_{x=0}^{x=1} y^2\, dx\, dz = -\int_{z=0}^{z=1} [x]_0^1\, dz \quad (\because y = 1 \text{ on } S_4)$$

$$= -[z]_0^1 = -1$$

$$\iint_{S_5} \vec{F}.\hat{n}\, ds_5 = \int_{z=0}^{z=1} \int_{y=0}^{y=1} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{i}\right) dy\, dz$$

$$=\int_{z=0}^{z=1}\int_{y=0}^{y=1} -4xz\,dy\,dz=0 \qquad \left(\sin ce\, x=0\ on\, S_5\right)$$

$$\iint_{S_6} \vec{F}\,.\hat{n}\,ds_6=\int_{z=0}^{z=1}\int_{y=0}^{y=1}\left(4xz\vec{i}-y^2\vec{j}+yz\vec{k}\right).\left(-\vec{i}\right)dy\,dz$$

$$=\int_{y=0}^{y=1}\int_{z=0}^{z=1} 4xz\,dy\,dz=4\int_{y=0}^{y=1}\left[\frac{z^2}{2}\right]_0^1 dy \qquad \left(\because\ x=1\ on\ S_6\right)$$

$$=\frac{4}{2}\,y=\int_{y=0}^{y=1}dy\ \ =\frac{4}{2}=2$$

$$\therefore\ RHS=\iint_S \vec{F}\,.\hat{n}\,ds=0+\frac{1}{2}+0-1+0+2=\frac{3}{2}$$

$$\therefore\ LHS=RHS$$

13. a)i) Prove that $u=e^x\left(x\cos y-y\sin y\right)$ is harmonic and hence find the analytic function

$$f\left(z\right)=u+iv$$

Solution:

$$u=e^x\left(x\cos y-y\sin y\right)$$

$$\frac{\partial u}{\partial x}=e^x\left(x\cos y-y\sin y\right)+e^x\left(\cos y\right)$$

$$\frac{\partial^2 u}{\partial x^2}=e^x\left(x\cos y-y\sin y\right)+e^x\left(\cos y\right)+e^x\cos y$$

$$\frac{\partial u}{\partial y}=e^x\left(-x\sin y-\sin y-y\cos y\right)$$

$$\frac{\partial^2 u}{\partial y^2}=e^x\left(-x\cos y-\cos y-\cos y+y\sin y\right)$$

$$\frac{\partial^2 u}{\partial^{x^2}}+\frac{\partial^2 u}{\partial y^2}=e^x x\cos y-e^x y\sin y+2e^x\cos y-xe^x\cos y-2e^x\cos y+e^x y\sin y=0$$

$\therefore\ u$ is harmonic.

$$f^1(z)=\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

$$f^1(z)=\frac{\partial u}{\partial x}(z,0)-i\frac{\partial u}{\partial y}(z,0)$$

$$\frac{\partial u}{\partial x}(z,0)=ze^z+e^z \; ; \; \frac{\partial u}{\partial y}(z,0)=0$$

$$\therefore f^1(z)=ze^z+e^z$$

$$f(z)=\int\left(ze^z+e^z\right)dz+c$$

$$=ze^z-e^z+e^z+c$$

$$f(z)=ze^z+c$$

13.a) ii) Find the bilinear transformation that transforms 1, 1 and -1 in the z-plane on to 0, 1 and ∞ of the w-plane. Also show that the transformation maps interior of the unit circle of the z-plane on to upper half of the w-plane.

Solution:

By invariance property of cross ration

$$\frac{(w-w_1)(w_3-w_2)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_3-z_2)}{(z-z_3)(z_2-z_1)} \qquad z_1=1, z_2=i, z_3=1$$

$$\frac{(w-w_1)w_3\left(1-\frac{w_2}{w_3}\right)}{-w_3\left(1-\frac{w}{w_3}\right)(w_2-w_1)}=\frac{(z-1)(-1-i)}{(z+1)(i-1)} \qquad w_1=0, w_2=1, w_3=\infty$$

$$\frac{w}{-1}=\frac{(1-z)(1+i)}{-(1+z)(1-i)}$$

$$w=\frac{(1-z)}{(1+z)}\quad\frac{(1+i)^2}{2}$$

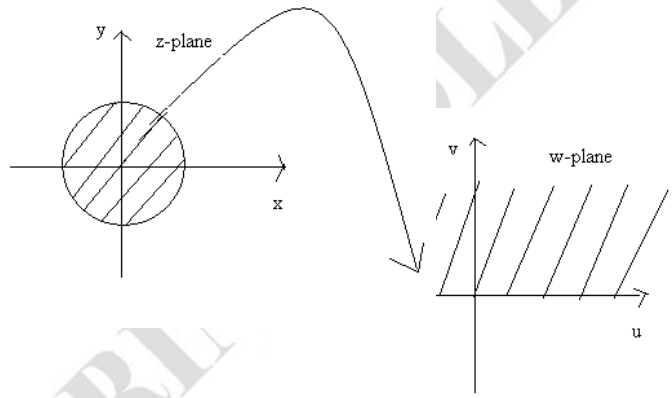
$$=\frac{(1-z)}{2(1+z)}(1-1+2i)=\frac{2i(1-z)}{2(1+z)}$$

$$w=\frac{i(1-z)}{(1+z)}$$

$$w(1+z)=i-iz \Rightarrow zw+iz=i-w \Rightarrow z=\frac{i-w}{w+i}$$

$$|z|<1$$

$$\left|\frac{i-w}{w+i}\right|<1 \qquad |i-w|<|w+i|$$



$$|i-u-iv| < |u+iv+i|$$

$$|-u+i(1-v)| < |u+i(i+v)|$$

$$u^2+(1-v)^2 < u^2+(1+v)^2$$

$$-2v < 2v$$

$$\Rightarrow 4v > 0 \Rightarrow v > 0$$

$\therefore$interior of the unit circle maps on to upper half of the z-plane.

13. b) i) Prove that $\mathrm{u}=\mathrm{x}^2-\mathrm{y}^2$ and $v=\frac{-y}{x^2+y^2}$ are harmonic but $\mathrm{u}+\mathrm{iv}$ is not regular.

Solution:

$$u=x^2-y^2;\ v=\frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=2x;\frac{\partial u}{\partial y}=-2y;\frac{\partial^2 u}{\partial x^2}=2;\frac{\partial^2 u}{\partial y^2}=-2$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \therefore u \text{ is harmonic}$$

$$\frac{\partial v}{\partial x}=\frac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)}=\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{(x^2+y^2)(2y)-2xy\ \ 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$=\frac{2yx^2+2y^3-8x^2y}{(x^2+y^2)^3}\ =\frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y}=\frac{(-1)(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2}=\frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2}=\frac{(2y)(x^2+y^2)^2-(y^2-x^2)\,2(x^2+y^2)(2y)}{(x^2+y^2)^3}$$

$$=\frac{2x^2y+2y^3-4y^3+4x^2y}{(x^2+y^2)^3}=\frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

$\therefore \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad \therefore u$ is harmonic

$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}\neq-\frac{\partial u}{\partial y}$

$\therefore\ u+iv$ is not regular (or analytic).

13. b) ii) Find the image of the half plane $x>c$, $c>o$ under $w=\frac{1}{z}$ sketch graphically. Also find the fixed points of w.

Solution:

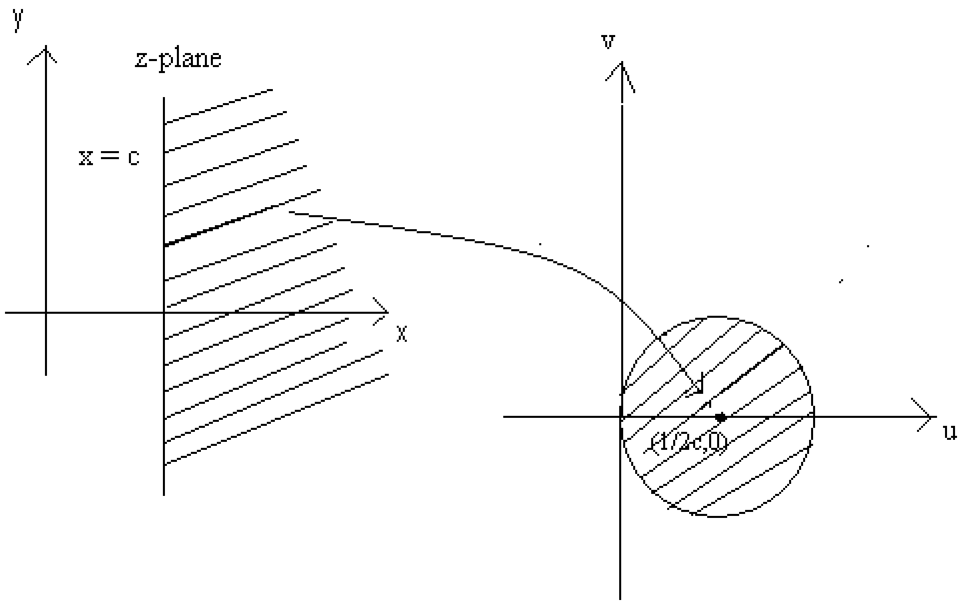
$w=\frac{1}{z}\Rightarrow z=\frac{1}{w} \quad ie., x+iy=\frac{1}{u+iv}\frac{u-iv}{u-iv}$

$\therefore x+iy=\frac{u-iv}{u^2+v^2} \qquad \therefore x=\frac{u}{u^2+v^2}\ ,\ y=\frac{v}{u^2+v^2}$

$x>c\ \Rightarrow\frac{u}{u^2+v^2}>c\Rightarrow\frac{u}{c}>u^2+v^2\Rightarrow u^2+v^2-u/c<0$

$ie.,\ \left(u-\frac{1}{2c}\right)^2+v^2<\left(\frac{1}{2c}\right)^2$

$\therefore$ image of the half plane $x>c$ is the interior of the circle in the w-plane with centre at $\left(\frac{1}{2c},0\right)$ and radius $\frac{1}{2c}$



Fixed points of $w = \frac{1}{z}$ are $z^2 = 1$ *i.e.*, $z = \pm 1$.

14. a) i) Evaluate $\int \frac{(z+4)\,dz}{z^2+2z+5}$, where C is the circle $|z+1+i| = z$ using Cauchy's integral formula.

Solution:

Singularities of the function $\frac{z+4}{z^2+2z+5}$ are $z^2+2z+5=0$

ie., $z = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm 4i}{2} = -4 \pm 2i$ of these singularities , $-1-2i$ lies within $|z+1+i| = z$ and $-1-2i$ lies outside

$$\therefore \int_C \frac{(z+4)\,dz}{z^2+2z+5} = \int_C \frac{\frac{(z+4)}{(z-(-1+2i))}}{z-(-1-2i)}\; dz$$

By Cauchy's integral formula

$$\int_C \frac{\left(\frac{z+1}{z+1-2i}\right) dz}{(z-(-1-2i))} = 2\pi i\, f(-1-2i)$$

where $f(z) = \frac{z+1}{z+1-2i}$

$$\therefore f(-1-2i) = \frac{-1-2i+1}{-1-2i+1-2i} = \frac{-2i}{-4i} = \frac{1}{2}$$

$$\therefore \int_C \frac{(z+4)\,dz}{z^2+2z+5} = 2\pi i \times \frac{1}{2} = \pi i$$

14. a) ii) Find the residues of $f(z)=\dfrac{z^2}{(z-1)^2(z+2)^2}$ at its isolated singularities using Laurent's series expansions. Also state the valid region.

Solution:

$$f(z)=\frac{z^2}{(z-1)^2(z+2)^2}=\frac{(z-1+1)^2}{(z-1)^2(z-1+3)^2}=\left[\frac{(z-1)^2+1+2(z-1)}{(z-1)^2}\right]\frac{1}{9\left(1+\left(\frac{z-1}{3}\right)\right)^2}$$

$$=\frac{1}{9}\left[1+\frac{1}{(Z-1)^2}+\frac{2}{(Z-1)}\right]\left[1+\left(\frac{Z-1}{3}\right)\right]^{-2}$$

$$=\frac{1}{9}\left[1+\frac{1}{(Z-1)^2}+\frac{2}{(Z-1)}\right]\left[1-2\left(\frac{z-1}{3}\right)+3\left(\frac{z-1}{3}\right)^2-\cdots 0\right]\quad (\text{valid when } |z-1|<3)$$

$$=\frac{1}{9}\left[\frac{2}{(z-1)}-2/3\,\frac{1}{(z-1)}+\cdots\cdots\infty\right]$$

$\therefore$ Res of $f(z)$ at $z=1$ is the coefficient of $\dfrac{1}{z-1}$ in the Laurent's series expansion

$$=\frac{1}{9}\left[2-2/3\right]=1/9\times 4/3=4/27 \text{ valid in the region } 0<|z-1|<3$$

$$f(z)=\frac{z^2}{(z-1)^2(z+2)^2}=\frac{(z+2-2)^2}{(z+2-3)^2(z+2)^2}$$

$$=\left[\frac{(z+2)^2+4-4(z+2)}{(z+2)^2}\right]\frac{1}{9\left(1-\left(\frac{z+2}{3}\right)\right)^2}]$$

$$=\frac{1}{9}\left[1+\frac{4}{(z+2)^2}\;\frac{-4}{(z+2)}\right]\left(1-\left(\frac{z+2}{3}\right)\right)^{-2}$$

$$=\frac{1}{9}\left[1-\frac{4}{(z+2)^2}\;\frac{-4}{(z+2)}\right]\left(1+2\left(\frac{z+2}{3}\right)+3\left(\frac{z+2}{3}\right)^2+\cdots\infty\right)$$

$$=\frac{1}{9}\left[\frac{-4}{(z+2)}+8/3\;\frac{1}{(z+2)}+\cdots\infty\right]\quad (\text{valid in the region } |z+2|<3)$$

$\therefore$ Res of $f(z)$ at $z=-2$ is the coefficient of $\frac{1}{z+2}$ in the Laurent's series expansion

$=\frac{1}{9}\left[-4+\frac{8}{3}\right]=\frac{-4}{27}$ valid in the region $0<|z+2|<3$.

14. b) i) Evaluate $\int_0^{\pi}\frac{d\theta}{2+\cos\theta}$

(Refer solution of **14. b) i) in V 4012**)

14. b) ii) Evaluate $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}$ using contour integration.

Solution:

Consider the integral $\int_C \frac{dz}{(z^2+1)(z^2+4)}$ where C is a closed curve consisting of the semicircle C_1 with centre at origin and radius R which is large enough such that all the singularities of $\frac{1}{(z^2+1)(z^2+4)}$ above the real axis line within the semicircle and the line segment along the real axis joining –R to R.

$$\int_C \frac{dz}{(z^2+1)(z^2+4)}=\int_{C_1}\frac{dz}{(z^2+1)(z^2+4)}+\int_{-R}^{R}\frac{dx}{(x^2+1)(x^2+4)} \quad \text{(along the real line } z=x\text{)}$$

$$\int_C \frac{dz}{(z^2+1)(z^2+4)}=2\pi i\left[\left(\operatorname{Res} f(z)_{z=i}+\left(\operatorname{Res} f(z)_{z=2i}\right)\right)\right] \text{(by Cauchy residue theorem)}$$

Where $f(z)=\frac{1}{(z^2+1)(z^2+4)}$ (singularities of $f(z)$ are $z=i,-i,2i,-2i$

$$\left[\operatorname{Res} f(z)\right]_{z=i}=lt_{z\to i}(z-i)\frac{1}{(z-i)(z+i)(z^2+4)}=\frac{1}{2i(-1+4)}=\frac{1}{6i}$$

$$\left[\operatorname{Res} f(z)\right]_{z=2i}=lt_{z\to 2i}(z-2i)\frac{1}{(z-2i)(z+2i)(z^2+1)}=\frac{1}{(4i)(-3)}=\frac{-1}{12i}$$

$$\therefore \int_C \frac{dx}{(z^2+1)(z^2+4)}=2\pi i\left[\frac{1}{6i}-\frac{1}{12i}\right]=2\pi i\left[\frac{1}{12i}\right]=\pi/6$$

$$\therefore \frac{\pi}{6}=\int_{C_1}\frac{dz}{(z^2+1)(z^2+4)}+\int_{-R}^{R}\frac{dx}{(x^2+1)(x^2+4)}$$

Applying $\lim t_{R\to\infty}$

$$\frac{\pi}{6} = lt_{R\to\infty} \int_{C_1} \frac{dz}{(z^2+1)(z^2+4)} + \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} \qquad \text{----------} (1)$$

$$\left|\int_{C_1} \frac{dz}{(z^2+1)(z^2+4)}\right| \le \int_{C_1} \frac{|dz|}{|z^2+1||z^2+4|}$$

on C_1, $z = \mathrm{Re}^{i\theta}$

$dz = i\,\mathrm{Re}^{i\theta}\, d\theta$

$|dz| = R d\theta$

$0 \le \theta \le \pi$

$$\le \int_{o}^{\pi} \frac{R d\theta}{(|z|^2-1)(|z|^2-4)}$$

$$\le \frac{R}{(R^2-1)(R^2-4)} \int_0^{\pi} d\theta$$

$$\le \frac{\frac{R}{R^4}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} \pi$$

$$lt_{R\to\infty} \left|\int_{C_1} \frac{dz}{(z^2+1)(z^2+4)}\right| \le lt_{R\to 0} \frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} = 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$$

15. a) i) Find the Laplace transform of $f(t) \begin{cases} E & 0 \le t \le a \\ -E & a \le t \le 2a \end{cases}$ and $f(t+2a) = f(t)$ for all t.

Solution:

$$L\{f(t)\} = \frac{1}{1-e^{-2as}} \left[\int_0^a E\, e^{-st} dt + \int_0^a -E\, e^{-st} dt\right] = \frac{E}{1-e^{-2as}} \left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{E}{1-e^{-2as}} \left[\left(\frac{e^{-2as}-1}{-s}\right) - \left(\frac{e^{-2as}-e^{-as}}{-s}\right)\right]$$

$$= \frac{E}{1-e^{-2as}} \left[\left(\frac{e^{-2as}+1-2e^{-as}}{s}\right)\right]$$

$$= \frac{E}{\left(1-e^{-as}\right)\left(1+e^{-as}\right)} \frac{\left(1-e^{as}\right)^2}{s}$$

$$= \frac{E}{s} \frac{\left(1-\frac{e^{-\frac{as}{2}}}{e^{\frac{as}{2}}}\right)}{\left(1+\frac{e^{-\frac{as}{2}}}{e^{\frac{as}{2}}}\right)} = \frac{E}{s} \frac{\frac{\left(e^{\frac{as}{2}}-e^{-\frac{as}{2}}\right)}{2}}{\frac{\left(e^{\frac{as}{2}}+e^{-\frac{as}{2}}\right)}{2}}$$

$$= \frac{E}{s} \frac{\sinh\frac{as}{2}}{\cosh\frac{as}{2}} = \frac{E}{s}\tanh\frac{as}{2}$$

.

15.a) ii) Find the inverse Laplace transform of $\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$ using convolution theorem.

Solution:

$$L^{-1}\left\{\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\,\frac{s}{s^2+b^2}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2+a^2}\right\} \times L^{-1}\left\{\frac{s}{s^2+b^2}\right\} \qquad \text{(by convolution theorem)}$$

$$= \cos at \times \cos bt = \int_{u=0}^{t} \cos au\,\cos b\left(t-u\right)$$

$$= \frac{1}{2}\int_{u=0}^{t}\left[\cos\left(au+bt-bu\right)+\cos\left(au-bt+bu\right)\right] du$$

$$= \frac{1}{2}\left[\frac{\sin\left(\left(a-b\right)u+bt\right)}{\left(a-b\right)} + \frac{\sin\left(\left(a+b\right)u-bt\right)}{\left(a+b\right)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin\left(\left(a-b\right)t+bt\right)}{\left(a-b\right)} + \frac{\sin\left(\left(a+b\right)t-bt\right)}{\left(a+b\right)}\right] - \left(\frac{\sin bt}{\left(a-b\right)} + \frac{\sin\left(-bt\right)}{\left(a+b\right)}\right)$$

$$= \frac{1}{2}\left[\left(\frac{\sin at}{\left(a-b\right)} + \frac{\sin at}{\left(a+b\right)}\right) + \left(\frac{\sin bt}{\left(a+b\right)} + \frac{\sin bt}{\left(a-b\right)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right] = \frac{a\sin at}{a^2-b^2} - \frac{b\sin bt}{a^2-b^2}$$

.

15. b) i) Verify initial and final value theorems for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

Solution:

<u>Initial Value Theorem</u>

$$\lim_{t\to 0} f(t) = lt_{s\to\infty} sF(s)$$

$$F(s) = L\{f(t)\} = L\left\{1 + e^{-t}\sin t + e^{-t}\cos t\right\}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1}$$

$$LHS = lt_{t\to 0} f(t) = lt_{t\to 0} 1 + e^{-t}(\sin t + \cos t) \quad = 1 + 1 = 2$$

$$RHS = lt_{s\to\infty} sF(s) = lt_{s\to\infty} \; s\left[\frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}\right]$$

$$= lt_{s\to\infty}\left[1 + \frac{\frac{1}{s}}{1+\frac{2}{s}+\frac{2}{s^2}} + \frac{1+\frac{1}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1 + 0 + 1 = 2$$

$$\therefore LHS = RHS$$

<u>Final Value Theorem</u>

$$\lim_{t\to\infty} f(t) = lt_{s\to 0} \; sF(s)$$

$$LHS = lt_{t\to\infty} f(t) = lt_{t\to\infty} \; 1 + e^{-t}(\sin t + \cos t) = 1$$

$$RHS = lt_{s\to 0} sF(s) = lt_{s\to 0}\left[\frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}\right]$$

$$= \lim_{s\to 0} 1 + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{s^2+2s+2} \quad = 1$$

$$\therefore LHS = RHS$$

15.b) ii) Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$ with $y(0) = 1 = y'(0)$.

Solution:

$$L\{y''-3y'-4y\}=L\{2e^{-t}\}$$

$$L\{y''\}-3L\{y'\}-4L\{y\} = \frac{2}{s+1}$$

$$\left[s^2L\{y\}1-\{sy(0)-y'(0)\}\right]-3\left[sL\{y\}-y(0)\right]-4L\{y\} = \frac{2}{s+1}$$

$$\left(s^2-3s-4\right)L\{y\} - s-1+3 = \frac{2}{s+1}$$

$$\left(s^2-3s-4\right)L\{y\}=\frac{2}{s+1}+s-2$$

$$L\{y\}=\frac{2}{(s+1)^2(s-4)}+\frac{s-2}{(s-4)(s+1)}$$

$$\therefore y=L^{-1}\left\{\frac{2}{(s-4)(s+1)^2}+\frac{s-2}{(s-4)(s+1)^2}\right\}$$

$$L^{-1}\left\{\frac{2}{(s-4)(s+1)^2}\right\} = L^{-1}\left\{\frac{A}{s+1}+\frac{B}{s-y}+\frac{c}{(s+1)^2}\right\}$$

To find the values of A, B, C

$$\frac{2}{(s+1)(s-4)(s+1)} = \frac{A}{s+1}+\frac{B}{s-4}+\frac{C}{(s+1)^2}$$

$$2 = A(s+1)(s-4)+B(s+1)^2+C(s-4)$$

$$put\ s=-1, \quad 2=-5C \Rightarrow C = {-2}/{5}$$

$$put\ s=4, \quad 2=25B \Rightarrow B = {2}/{25}$$

$$put\ s=0, \quad 2=-4A+B-4C$$

$$2=-4A+\frac{2}{25}+\frac{8}{5} \Rightarrow -4A=2-\frac{42}{25}$$

$$-4A=\frac{8}{25}$$

$$\Rightarrow A={-2}/{25}$$

$$\therefore L-1\left\{\frac{2}{(s-4)(s+1)^2}\right\} = \frac{-2}{25}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{2}{25}L^{-1}\left\{\frac{1}{s-4}\right\}\frac{-2}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$=\frac{-2}{25}e^{-t}+\frac{2}{25}e^{4t}-\frac{2}{5}e^{-t}t$$

$$L^{-1}\left\{\frac{(s-2)}{(s-4)(s+1)}\right\}=L^{-1}\left[\frac{A}{s-4}+\frac{B}{s+1}\right]$$

$$\frac{s-2}{(s-4)(s+1)}=\frac{A(s+1)+B(s-4)}{(s-4)(s+1)}$$

$$s-2=A(s+1)+B(s-4)$$

Put $s=-1,\ -3=-5B\ \Rightarrow B=\frac{3}{5}$

Put $s=4,\ 2=54\Rightarrow A=\frac{2}{5}$

$$\therefore L^{-1}\left[\frac{(s-2)}{(s-4)\,(s+1)}\right]=\frac{2}{5}L^{-1}\left[\frac{1}{s-4}\right]+\frac{3}{5}L^{-1}\left[\frac{1}{s+1}\right]$$

$$=\frac{2}{5}e^{4t}+\frac{3}{5}e^{-t}$$

$$\therefore y=L^{-1}\left[\frac{2}{(s-4)(s+1)^2}\right]+L^{-1}\left[\frac{(s-2)}{(s-4)(s+1)}\right]$$

$$=\frac{-2}{25}e^{-t}+\frac{2}{25}e^{4t}-\frac{2}{5}e^{-t}t+\frac{2}{5}e^{4t}+\frac{3}{5}e^{-t}$$

$$=\frac{13e^{-t}}{25}+\frac{12}{25}e^{4t}-\frac{2}{5}te^{-t}$$